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DEPARTMENTS

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Pseudo logarithms

HAROLD D. LARSEN, *Albion College, Albion, Michigan.*

The drive to shorten the labor required to do tedious calculations is recorded by attempts to develop methods that did not stand the test of time.

JOHN NAPIER invented logarithms in 1614. These valuable mathematical tools owe their great importance to the fact that they enable us to simplify certain arithmetical computations. In particular, logarithms permit the computer to replace the operation of multiplication by the relatively easier operation of addition. However, logarithms are not the only quantities which lead to such simplifications. Indeed, there are several quantities which may be used to reduce a multiplication to additions and subtractions. We shall call such quantities *pseudo logarithms;* four examples of pseudo logarithms are discussed in this article.

PROSTHAPHAERESIS

A table of natural sines can be used to replace a product by four subtractions, one addition, and a division by 2. Such a method was devised by Wittich of Breslau in the latter half of the sixteenth century. For a short time Wittich was an assistant to the great Danish astronomer Tycho Brahe (1546–1601), and together they used this method as early as 1582 to carry out lengthy astronomical multiplications. The method disappeared upon the invention of logarithms.

The method of Wittich, called *prosthaphaeresis,* is based on the following trigonometrical identity:

$$\sin A \sin B = \tfrac{1}{2}(\sin P - \sin Q)$$

where

$$P = 90^\circ - (A - B), \quad Q = 90^\circ - (A + B).$$

Four entries in a table of natural sines are required to compute a product with the aid of this formula. The following example in which a five-place table is used illustrates the method.

EXAMPLE:

$$R = (0.8695)(0.3170) = ?$$

SOLUTION:

$$A = \text{Arc sin } (0.8695) = 60^\circ 24.1'$$
$$B = \text{Arc sin } (0.3170) = 18^\circ 28.9'$$
$$A - B = 41^\circ 55.2'$$
$$A + B = 78^\circ 53.0'$$
$$P = 90^\circ - (A - B) = 48^\circ 4.8'$$
$$Q = 90^\circ - (A + B) = 11^\circ 7.0'$$
$$\sin P = 0.74408$$
$$\sin Q = \underline{0.19281} -$$
$$2\overline{)0.55127}$$
$$R = 0.27563.$$

Obviously, in applying the method of prosthaphaeresis, each factor in a multiplication must first be reduced to a number between 0 and 1 before a table of natural sines can be used. For example,

$$\begin{aligned}(869.5)(3.170) &= (0.8695 \times 10^3)(0.3170 \times 10^1)\\ &= (0.8695 \times 0.3170)(10^3 \times 10^1)\\ &= (0.27563) \times (10^4)\\ &= 2{,}756.3\end{aligned}$$

We see here an analogy to the characteristics of common logarithms.

It has been speculated that prosthaphaeresis may have given John Napier the inspiration for his logarithms.

Squares

Among the cuneiform texts left to us by the ancient Babylonians are many of mathematical interest. These clay tablets reveal that the Babylonians were remarkably well advanced in their knowledge of mathematics. Many of the cuneiform texts are numerical tables of one sort or another, with tables of squares rather common. Tables of squares are not very exciting in themselves, but it has been suggested recently that the Babylonians might have used them as pseudo logarithms. Indeed, a table of squares can be used to reduce a multiplication to the easier operations of one addition, one subtraction, and two divisions by 2.

The following example illustrates one way of using squares to calculate $m \cdot n$ when m and n are both even or are both odd. (You might like to investigate yourself the procedure to be followed when one factor is odd and the other is even.) The method is based on this fact. If

$$x=\tfrac{1}{2}(m+n) \quad \text{and} \quad y=\tfrac{1}{2}(m-n),$$

then

$$m=x+y, \qquad n=x-y,$$

and

$$m \cdot n=x^2-y^2.$$

Example 1:

$$(19)(13)=?$$

Solution:

$$x=\tfrac{1}{2}(19+13)=16, \quad y=\tfrac{1}{2}(19-13)=3$$
$$(19)(13)=16^2-3^2=256-9=247.$$

Example 2:

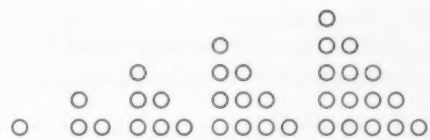
$$(1.99)(0.023)=?$$

Solution:

$$\begin{aligned}(1.99)(0.023)&=(199\times10^{-2})(23\times10^{-3})\\&=(199\times23)(10^{-5})\\&=(111^2-88^2)(10^{-5})\\&=(12{,}321-7{,}744)(10^{-5})\\&=(4{,}577)(10^{-5})=0.04577.\end{aligned}$$

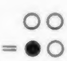
Triangular numbers

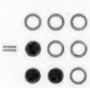
Pythagoras and his school developed the practice of representing numbers by geometric figures. The study of triangular, square, pentagonal, and other *figurate* numbers was once considered an important chapter of algebra, having practical applications to problems involving piles of cannon balls. The Pythagoreans made use of figurate numbers to discover many important formulas of arithmetic. To illustrate, consider the following triangular numbers.

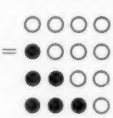


$T(1)$ $T(2)$ $T(3)$ $T(4)$ $T(5)$

By simple juxtaposition,

$T(1)+T(2)=$  $=2^2$,

$T(2)+T(3)=$  $=3^2$,

$T(3)+T(4)=$  $=4^2$,

and, in general,

$$T(n-1)+T(n)=n^2.$$

A formula for the nth triangular number is derived rather easily. For, clearly,

$$T(n)=1+2+3+\cdots+(n-2)+(n-1)+n,$$

or, adding in reverse order,

$$T(n)=n+(n-1)+(n-2)+\cdots+3+2+1.$$

Adding these two equations term by term, we obtain

$$2T(n) = (n+1)+(n+1)+ \cdots$$
$$+(n+1) \text{ to } n \text{ terms}$$
$$= n(n+1),$$

whence

$$T(n) = \tfrac{1}{2}n(n+1).$$

Another formula is very useful. We may write

$$T(n+1)$$
$$= 1+2+3+ \cdots +(n-1)+n+(n+1)$$
$$= [1+2+3+ \cdots +(n-1)+n]+(n+1)$$
$$= T(n)+(n+1).$$

The latter formula permits the construction of a table of triangular numbers by continuous addition. The table of triangular numbers given in Table 1 was computed in this manner.

The identity

$$m \cdot n = T(m+n) - T(m) - T(n)$$

is easy to derive and the proof is left to the reader. With the aid of this identity, a table of triangular numbers can be enlisted to serve as pseudo logarithms. Table 1 suffices to obtain the product of any two two-figure numbers.

Example 1:

$$(73)(59) = T(132) - T(73) - T(59)$$
$$= 8{,}778 - 2{,}701 - 1{,}770$$
$$= 4{,}307.$$

Example 2:

$$(0.038)(0.45) = (38)(45)(10^{-5})$$
$$= [T(83) - T(38) - T(45)]10^{-5}$$
$$= (3{,}486 - 741 - 1{,}035) \cdot 10^{-5}$$
$$= (1{,}710) \cdot 10^{-5} = 0.01710.$$

Quarter squares

At one time quarter squares actually were called logarithms. The first practical application of these pseudo logarithms was made in 1817 by Antoine Voisin, who published a table of quarter squares for all integers less than 20,000. Voisin's

TABLE 1

Triangular numbers

N	0	1	2	3	4	5	6	7	8	9
0	0	1	3	6	10	15	21	28	36	45
1	55	66	78	91	105	120	136	153	171	190
2	210	231	253	276	300	325	351	378	406	435
3	465	496	528	561	595	630	666	703	741	780
4	820	861	903	946	990	1,035	1,081	1,128	1,176	1,225
5	1,275	1,326	1,378	1,431	1,485	1,540	1,596	1,653	1,711	1,770
6	1,830	1,891	1,953	2,016	2,080	2,145	2,211	2,278	2,346	2,415
7	2,485	2,556	2,628	2,701	2,775	2,850	2,926	3,003	3,081	3,160
8	3,240	3,321	3,403	3,486	3,570	3,655	3,741	3,828	3,916	4,005
9	4,095	4,186	4,278	4,371	4,465	4,560	4,656	4,753	4,851	4,950
10	5,050	5,151	5,253	5,356	5,460	5,565	5,671	5,778	5,886	5,995
11	6,105	6,216	6,328	6,441	6,555	6,670	6,786	6,903	7,021	7,140
12	7,260	7,381	7,503	7,626	7,750	7,875	8,001	8,128	8,256	8,385
13	8,515	8,646	8,778	8,911	9,045	9,180	9,316	9,453	9,591	9,730
14	9,870	10,011	10,153	10,296	10,440	10,585	10,731	10,878	11,026	11,175
15	11,325	11,476	11,628	11,781	11,935	12,090	12,246	12,403	12,561	12,720
16	12,880	13,041	13,203	13,366	13,530	13,695	13,861	14,028	14,196	14,365
17	14,535	14,706	14,878	15,051	15,225	15,400	15,576	15,753	15,931	16,110
18	16,290	16,471	16,653	16,836	17,020	17,205	17,391	17,578	17,766	17,955
19	18,145	18,336	18,528	18,721	18,915	19,110	19,306	19,503	19,701	19,900

table could be used to multiply all numbers of four significant figures. A more extensive table was published by Joseph Blater in 1887. Blater's "Table of Quarter Squares of All Whole Numbers from 1 to 200,000" permits the multiplication of all numbers of five significant figures.

The method of quarter squares is based on a simple algebraic identity,

$$m \cdot n = \tfrac{1}{4}(m+n)^2 - \tfrac{1}{4}(m-n)^2.$$

(We do not deprive the reader of the pleasure of deriving this identity himself.)

It is also easy to show that, if m and n are any two integers, then $(m+n)$ and $(m-n)$ are either both even or both odd. Now, if they are both even, then

$$\tfrac{1}{4}(m+n)^2 - \tfrac{1}{4}(m-n)^2$$
$$= \tfrac{1}{4}(2a)^2 - \tfrac{1}{4}(2b)^2 = a^2 - b^2.$$

On the other hand, if $(m+n)$ and $(m-n)$ are both odd, then

$$\tfrac{1}{4}(m+n)^2 - \tfrac{1}{4}(m-n)^2$$
$$= \tfrac{1}{4}(2a+1)^2 - \tfrac{1}{4}(2b+1)^2$$
$$= (a^2+a) - (b^2+b).$$

Thus, the fractions $\tfrac{1}{4}$ disappear from the difference and may be ignored safely.

We are now ready to define the *quarter square* of a number N, which will be designated by the symbol $Q(N)$:

$$\text{If } N = 2a, \qquad Q(N) = a^2;$$

$$\text{If } N = 2a+1, \qquad Q(N) = a^2 + a.$$

With this definition, the fundamental identity above takes the form,

$$m \cdot n = Q(m+n) - Q(m-n).$$

In words, *the product of two integers is equal to the quarter square of their sum diminished by the quarter square of their difference.*

If the quarter squares of $(m+n)$ and $(m-n)$ are readily available from tables, it is seen that the product of m and n can be determined by an addition and two subtractions.

A table of quarter squares may be constructed in a continuous manner. The construction is based on the following simple algebraic considerations.

TABLE 2

QUARTER SQUARES

N	0	1	2	3	4	5	6	7	8	9
0	0	0	1	2	4	6	9	12	16	20
1	25	30	36	42	49	56	64	72	81	90
2	100	110	121	132	144	156	169	182	196	210
3	225	240	256	272	289	306	324	342	361	380
4	400	420	441	462	484	506	529	552	576	600
5	625	650	676	702	729	756	784	812	841	870
6	900	930	961	992	1,024	1,056	1,089	1,122	1,156	1,190
7	1,225	1,260	1,296	1,332	1,369	1,406	1,444	1,482	1,521	1,560
8	1,600	1,640	1,681	1,722	1,764	1,806	1,849	1,892	1,936	1,980
9	2,025	2,070	2,116	2,162	2,209	2,256	2,304	2,352	2,401	2,450
10	2,500	2,550	2,601	2,652	2,704	2,756	2,809	2,862	2,916	2,970
11	3,025	3,080	3,136	3,192	3,249	3,306	3,364	3,422	3,481	3,540
12	3,600	3,660	3,721	3,782	3,844	3,906	3,969	4,032	4,096	4,160
13	4,225	4,290	4,356	4,422	4,489	4,556	4,624	4,692	4,761	4,830
14	4,900	4,970	5,041	5,112	5,184	5,256	5,329	5,402	5,476	5,550
15	5,625	5,700	5,776	5,852	5,929	6,006	6,084	6,162	6,241	6,320
16	6,400	6,480	6,561	6,642	6,724	6,806	6,889	6,972	7,056	7,140
17	7,225	7,310	7,396	7,482	7,569	7,656	7,744	7,832	7,921	8,010
18	8,100	8,190	8,281	8,372	8,464	8,556	8,649	8,742	8,836	8,930
19	9,025	9,120	9,216	9,312	9,409	9,506	9,604	9,702	9,801	9,900

$$Q(2k)-Q(2k-1)$$
$$=Q(2k)-Q(\overline{2k-1}+1)$$
$$=k^2-[(k-1)^2+(k-1)]$$
$$=k^2-(k^2-2k+1)-(k-1)$$
$$=k^2-k^2+2k-1-k+1=k.$$

Also,

$$Q(2k+1)-Q(2k)=k.$$

Thus,

when $k=1$, $Q(2)-Q(1)=1$

and $Q(3)-Q(2)=1$;

when $k=2$, $Q(4)-Q(3)=2$

and $Q(5)-Q(4)=2$, etc.

Therefore,

$$Q(2)=Q(1)+1,$$
$$Q(3)=Q(2)+1,$$
$$Q(4)=Q(3)+2,$$
$$Q(5)=Q(4)+2,$$

in which we add successively the numbers 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, and so on, as far as we wish to go.

Table 2, which was constructed in this fashion, permits the multiplication of any two numbers of two significant figures each.

EXAMPLE 1:

$$(42)(23)=Q(42+23)-Q(42-23)$$
$$=Q(65)-Q(19)$$
$$=1{,}056-90=966.$$

EXAMPLE 2:

$$(89{,}000)(7{,}500)=(89\times 10^3)(75\times 10^2)$$
$$=(89)(75)\times 10^5$$
$$=[Q(164-Q(14)]\times 10^5$$
$$=[6{,}724-49]\times 10^5$$
$$=667{,}500{,}000.$$

Have you read?

JOLY, ROXEE W. "A Lesson in Tenth-Year Mathematics," *High Points*, April 1958, pp. 23–28.

Here is an illustrative lesson which has as one of its purposes the teaching of the meaning of postulational thinking. It would be of no value to point out a part of the lesson. You must read it in its entirety. The lesson takes into consideration the proposition, postulate, theorems, and experimental thinking. I strongly advise reading this lesson, not because you will do the lesson, but because it will set you to thinking about ways of doing it better.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

LINDSAY, FRANK B., and others, "Science and Mathematics in California Public Schools," *California Schools*, April 1958, pp. 173–191.

This article summarizes mathematics and science in the California schools. The article contains many data, among which you will note the steady increase in mathematics offerings and in the per cent of students enrolled in these courses. For example, from 1948 to 1956 the increase in Algebra I was 48 per cent, in advanced Algebra 30 per cent, in Geometry 30 per cent, in Solid Geometry 70 per cent, and in Trigonometry 83 per cent.

From 1957 to 1958 the increase was: Algebra, 7 per cent; Advanced Algebra, 18 per cent; Geometry, 17 per cent; and Solid Geometry and Trigonometry together, 30 per cent. The increase in enrollment in college mathematics was 58 per cent for the same year.

This article gives similar percentages for science, and presents the subject areas taught in the elementary schools. You may want to compare these California statistics with the growth in your own schools.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Mathematics as a subject for learning plausible reasoning[1]

G. PÓLYA, *Stanford University, Palo Alto, California.*

Reasoning by analogy and intelligent guessing are important tools for the mathematician. Do teachers make enough use of them in mathematics classes?

MATHEMATICS is rightly esteemed as *the* subject for learning demonstrative reasoning. Here, however, I wish to advance a less usual thesis: *Mathematics is also an excellent subject for learning plausible reasoning, and should be exploited as such in our secondary schools.*

Much more space than is here available would be necessary to establish this thesis properly.[2] Here there is space only for some remarks on the nature of plausible reasoning, mathematical research, and mathematical instruction.

1. *Plausible reasoning.* We must distinguish between two types of reasoning: demonstrative and plausible. *Demonstrative reasoning* is inherent in mathematics and in pure logic; in other branches of knowledge it enters only insofar as the ideas in question seem to be raised to the logico-mathematical sphere. Demonstrative reasoning brings order and coherence to our conceptual systems, and it is therefore indispensable in the development of knowledge, but it cannot supply us with any new knowledge of the world around us. Such knowledge can be obtained, in science as in everyday life, only through *plausible reasoning.* The inferences from analogy and the inductive proofs of natural scientists, the statistical arguments of economists, the documentary evidence of historians, and the circumstantial evidence of lawyers can reasonably lay claim to our confidence, and to a very high degree, under favorable circumstances. But they are not demonstrative; all such arguments are merely plausible.

Demonstrative reasoning possesses a degree of certainty that is unattainable by merely plausible reasoning. Demonstrative reasoning has the stiff rules of formal logic to follow. Plausible reasoning, on the other hand, seems not to be subject to any comparable fixed rules; it is characterized by a certain fluidity that is very hard to describe. Plausible reasoning, moreover, does not have the clear, finished, and generally recognized theoretical basis of formal logic.

2. *Plausible reasoning in mathematical research.* We have here to distinguish between two aspects of mathematics. Mathematics as a *finished science* appears quite otherwise than mathematics *in the making.* In finished mathematics only axioms, definitions, and rigorous demonstrative arguments should find a place. In mathematical research, however, in the unfinished mathematics growing in the head and under the pencil of a mathematician, great or small, it is a different matter: there we find blind groping, guesses, occasional false steps, and many merely

[1] Published originally in German in *Gymnasium Helveticum,* a journal for Swiss secondary schools, in January, 1956; it is printed here with the permission of *Gymnasium Helveticum.* The translation is by C. M. Larsen, San Jose State College, San Jose, California.

[2] For a full treatment of this matter see the author's work, *Mathematics and Plausible Reasoning* (Princeton, N. J.: Princeton University Press, 1954).

plausible arguments. Here is the point which is of critical importance for our thesis: *Merely plausible arguments play a decisive role in developing mathematics.* This observation is by no means new; indeed, Euler and Laplace have expressed it in other words. To illustrate the truth of the observation, however, it is not easy to find an example that requires only a little elementary knowledge, that can be briefly described, and that is at the same time sufficiently impressive. Perhaps the following example will meet some of these specifications.

Consider a fixed circle, and n movable points on the circle, the n vertices of a (movable) polygon inscribed in the circle. With what arrangement of the vertices does the polygon attain the greatest possible area? When the n vertices lie equally spaced around the circle and the polygon is regular. This answer is easy to foresee, and it is also easy to prove, even without calculus (see, for example, the work cited in footnote 2, volume 1, pages 127–8).

Analogy now suggests that we investigate the corresponding situation in space. Consider a fixed sphere and n movable points on the sphere, the n vertices of an inscribed (movable) polyhedron. With what arrangement of the n vertices does the polyhedron attain the greatest possible volume?

The general problem is difficult. If, however, n is the number of vertices of a regular polyhedron, that is, if $n=4$, 6, 8, 12, or 20, a natural conjecture becomes obvious to us through analogy: the regular polyhedron attains the maximum volume.

In the simplest case, $n=4$, this conjecture is easy to verify: among all tetrahedra inscribed in a fixed sphere, the regular tetrahedron has the greatest volume (see, for example, the work cited in footnote 2, vol. 1, p. 133, example 17, and its solution on p. 245). In the next case, $n=6$, the conjecture also can be confirmed without difficulty (by a somewhat different method): the regular octahedron yields the maximum volume.

How does our conjecture stand now? *In mathematical work as in natural science, a conjecture becomes strengthened through analogy and through verification in special cases.* After the foregoing work, therefore, our conjecture looks reasonable and hopeful.

Nevertheless, the conjecture is wrong. Indeed, in the next case, $n=8$, the volume of the inscribed cube is less than the maximum attainable volume. Imagine a regular hexagon inscribed in the equatorial circle of the sphere, and erect a pyramid on each side, with the apex in the north or south pole respectively. The volume of such an inscribed hexagonal double-pyramid (a solid with eight vertices) is to the volume of the cube inscribed in the same sphere as 9 is to 8.

We have learned two things: First, that reasonable, plausible arguments do have a place in mathematics. Second, that a conjecture that has been strengthened through reasonable, plausible arguments can nevertheless be false.

In order to acquire the mental outlook of an adult, one must have certain experiences. If you wish to become a mature mathematician, you must pass through some experiences similar to the foregoing.

3. *Plausible reasoning in mathematics instruction.* After we have first carried out the calculation of the area of a triangle in simple cases (right triangles and isosceles triangles), we come finally to the general formula giving the area of a triangle, A, in terms of the sides, a, b, and c:

$$A=\sqrt{s(s-a)(s-b)(s-c)},$$

where

$$s=(a+b+c)/2.$$

The derivation is not at all simple, and an honest doubt can easily arise in an intelligent student who thinks for himself: How can I be sure this formula is right? Let's test it!

For this purpose the teacher should ask some questions. (Ultimately, he should

bring his students to the point of spontaneously proposing such questions themselves.)

1. Do the dimensions agree?
2. The area depends on each of the three sides in the same way: is the above expression symmetrical in a, b, and c?
3. Does the formula yield the correct area for an equilateral triangle $(a=b=c)$?
4. Does the formula yield the correct area for an isosceles triangle $(a=b)$?
5. Does the formula yield the correct area for a right triangle $(a^2+b^2=c^2)$?
6. What happens if the triangle degenerates into a straight line, so that $a+b$ becomes equal to c? Is the formula right then, too?

All of these questions, it turns out without too much trouble, can be answered in the affirmative. That the student thereby comes to have confidence in the formula and becomes acquainted with its structure is indeed well and good; but something more lies back of all this—much more than I can explain in this article. I do want, however, to hint at a few of these things.

First of all, this procedure is suitable not only for school, but also for research.

A mathematician thinks that he has found a new mathematical law. The law is expressed by a formula. Possibly the mathematician has only guessed the formula. Or, to be sure, he may have derived the formula; but his derivation may have a weak point, an unfilled gap, an unproved assumption. The mathematician cannot have confidence in the new formula without doing something more. In such a situation he can often with profit make direct use of the procedure that we have just used with the simple, school-level example: he can make deductions from the questionable formula and test them by comparing them with already well-established facts.

A naturalist thinks that he has found a new law of nature. This law may have been suggested to him through a striking observation, through a noteworthy analogy, or through a new application of a well-known principle. Whatever the source may have been, a well-trained naturalist does not trust the new law without doing something more: he makes deductions from the law and tests them by comparing them with experience.

Now, with an unprejudiced look, we can perceive the analogy and understand the fundamental point: the procedure that we employed with our simple, school-level example is in essence a basic procedure of research in both mathematics and natural science—*induction*. It seems to me that this proves that students can learn inductive reasoning as part of a well-conducted mathematics lesson. But this is a highly significant special case of the thesis which I advanced at the start.

Ample indirect evidence points to a compelling urge on the part of everyone concerned to promote from year to year and eventually to graduate every student, presumably in compliance with the quota rather than on the basis of achievement alone.—*Alexander G. Korol*, Soviet Education for Science and Technology, p. 195.

The cardioid

ROBERT C. YATES, *College of William and Mary, Williamsburg, Virginia.*

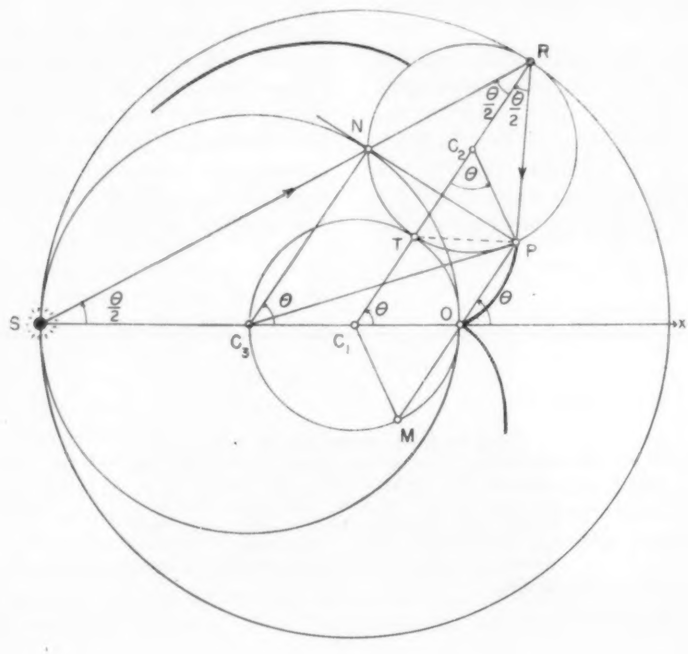
An investigation of the properties of a curve well known to mathematics teachers.

THE PROPERTIES of the cardioid are varied and surprising, its applications important, its classroom use well-nigh indispensable. Our purpose here is to display its character as a *roulette*, a *conchoid*, a *pedal*, a *caustic*, and an *envelope*, and then to determine its *arc length* and *area*, all in a manner that is entirely elementary. The time element alone would exclude it from the school program.

1. *Basic generation.* A circle, center C_2, radius a, rolls without slipping upon a fixed circle, center C_1, radius a (Figure 1). The cardioid is the locus of P, a point fixed on the rolling circle. Let $\overrightarrow{OP}$ represent the directed distance from O to P (where O is the point of contact of P with the fixed circle), and let θ represent the angle between $\overrightarrow{C_1OX}$ and $\overrightarrow{OP}$.

The line C_1C_2 of centers contains T, the point of tangency of the circles. Evidently, $\measuredangle TC_2P = \measuredangle OC_1T = \measuredangle XOP = \theta$, and OC_1C_2P is an isosceles trapezoid. The projection of its other three sides onto C_1C_2 gives the length relation

Figure 1



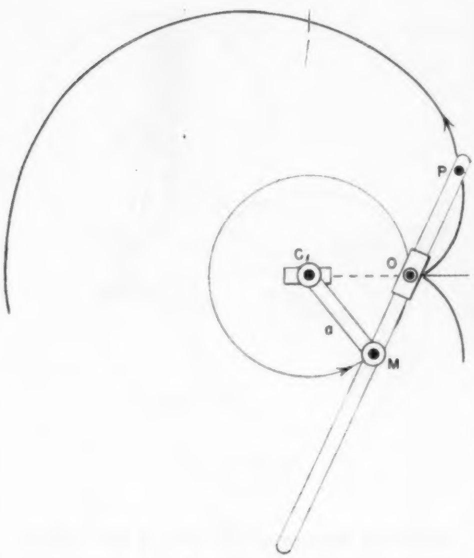


Figure 2

$$2a \cos\theta + \overrightarrow{OP} = 2a \quad \text{or} \quad \overrightarrow{OP} = 2a(1-\cos\theta).$$

2. *Conchoid.* Extend PO to meet the fixed circle at M. Then MC_1O is isosceles and MC_1C_2P is a parallelogram with $MP = MO + OP = C_1C_2 = 2a$; i.e., $2a \cos\theta + \overrightarrow{OP} = 2a$ as in (1). Thus the cardioid may be generated mechanically with the arrangement of Figure 2. The straight bar $PM(=2a$ in length) slides through a swivel slot fixed at O.

3. *Pedal.* In Figure 1, draw C_3N parallel to C_1T and the perpendicular to it from P. Then, projecting C_3O and OP onto C_3N,

$$\begin{aligned} C_3N &= 2a \cos\theta + OP \\ &= 2a \cos\theta + 2a(1-\cos\theta) = 2a. \end{aligned}$$

Accordingly, N lies always on a circle, center C_3, radius $2a$. The line NP is its tangent and P is the foot of the perpendicular to this tangent from O (Figure 3).

4. *Tangent line.* The point T of contact is the point of zero velocity (the instantaneous center of motion) of the rolling circle at the moment pictured in Figure 1. Any point P moves about T as a pivot, and P thus has direction of motion perpendicular to TP. Accordingly, the tangent to the cardioid at P is PR where R is diametrically opposite T, and $\measuredangle TRP = \theta/2$.

5. *Caustic.* In Figure 1, draw the circle, center C_1, radius $3a$, and extend OC_1 to meet this circle at S. Draw SR. Then, since

Figure 3

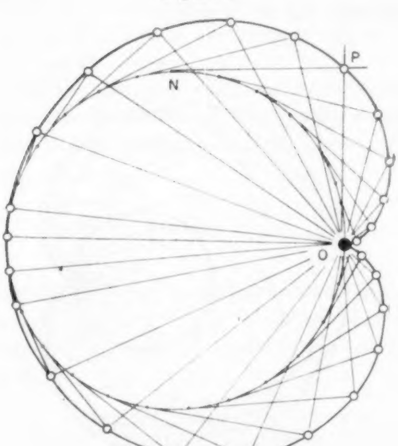
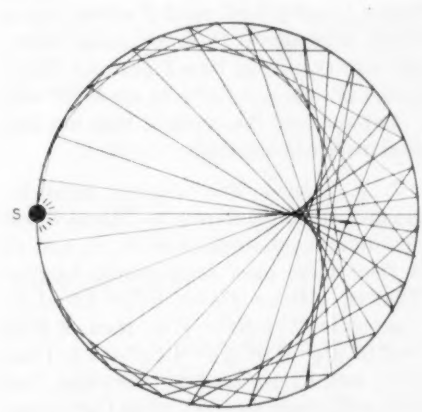


Figure 4



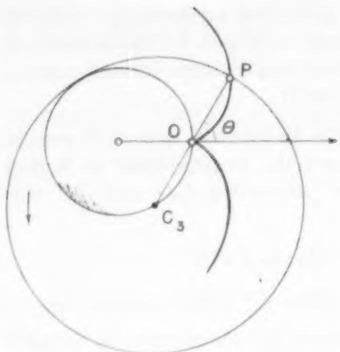


Figure 5

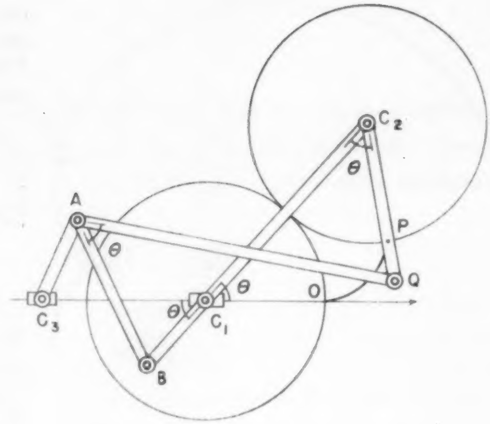


Figure 6

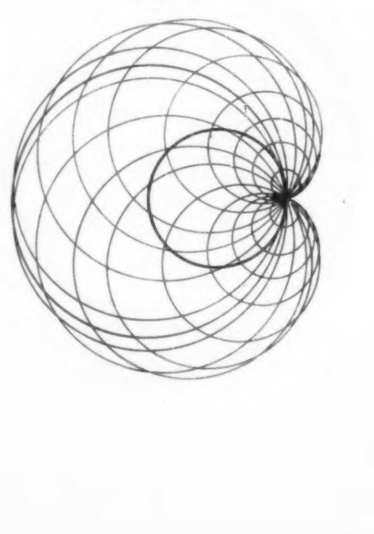
triangle SC_1R is isosceles, $\sphericalangle C_1SR = \sphericalangle C_1RS = \theta/2$. Accordingly, if SR be a ray of light emitted from S it is reflected by the circle at R along the ray RP, the tangent to the cardioid at P. With a radiant-point source on the circle, the cardioid is thus the envelope of rays reflected by the circle (Figure 4). This bright curve may be seen on the surface of liquid in a cup, the bottom of shiny pots, etc.

6. *Double generation.* Let the circle, center C_3, radius $2a$, now roll around the original fixed circle (Figure 5). The center C_3 would move into a position such as M in Figure 1, and a fixed point P of the rolling circle, originally at O, would move collinear with O and C_3. (See Figure 5.) Then, since $C_3P = 2a$ and $C_3O = 2a \cos\theta$, $OP = 2a - 2a\cos\theta$, and the cardioid thus has this dual mode of generation.

7. *A linkage.* Two crossed parallelograms jointed as shown in Figure 6 will maintain equal angles θ at C_1, A, and C_2 if their links have proportional lengths. That is, if $BC_2 = AQ = b$, $QC_2 = AB = C_1C_3 = a$, and $AC_3 = BC_1 = b^2/a$, then $\sphericalangle BAQ = \sphericalangle BC_1C_3 = \sphericalangle OC_1C_2 = \sphericalangle C_1C_2P = \theta$. Thus, if C_1 and C_3 be fixed to the plane, link C_1C_2 will rotate about C_1 while C_2P rotates oppositely about C_2 at the same rate. The action is then precisely that of the rolling circle described in (1), and P traces the cardioid.

8. *Envelope of circles.* Referring again to Figure 1, it is evident that $\widehat{OT} = \widehat{TP}$ and that the chord lengths TO and TP are therefore equal. Moreover, TP is perpendicular to PR, the tangent to the cardioid. Accordingly, the circle, center at T, passing through O, contains P and is tangent to PR. Thus the curve is enveloped (Figure 7) by the family of circles, with centers on the fixed circle and passing through one of its points.

Figure 7



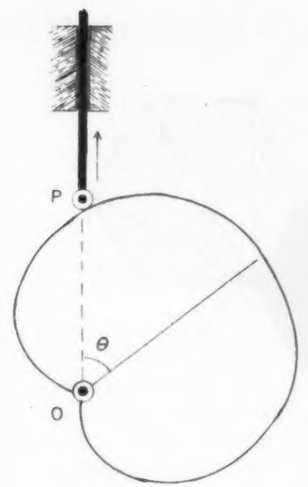


Figure 8

9. *Cam.* Since $\overrightarrow{OP}=2a(1-\cos\theta)$ or $\overrightarrow{OP}-2a=-2a\cos\theta$, the length $OP-2a$ changes harmonically if θ changes at a constant time rate. A cam, shaped as a cardioid, pivoted at the cusp (Figure 8) and rotating at constant angular velocity, thus produces harmonic motion in a pin guided in a straight slot and bearing on the cam.

10. *Length.* Consider two n-sided regular polygons, each with circumradius a, Figure 8. We fix one and roll the other upon it and attend to the locus of one of its vertices P.

As the polygon rolls, its vertices are successively centers of circles with diagonals r_k of the polygon as radii. These radii are

$$r_k=2a\sin k\cdot\frac{\pi}{n},$$

and the arcs they sweep out subtend angles $4\pi/n$. The sum of the $n-1$ arc lengths (until P returns to the starting point) is then

$$L_n=\frac{4\pi}{n}\sum_{k=1}^{n-1}r_k=\frac{8\pi a}{n}\sum_{k=1}^{n-1}\sin k\cdot\frac{\pi}{n}\,.$$

This expression may be consolidated by using the following identity:

$$\sum_{k=1}^{n-1}\sin k\theta\equiv\frac{\sin\dfrac{n}{2}\theta\sin\dfrac{n-1}{2}\theta}{\sin\dfrac{\theta}{2}}$$

to

$$L_n=\frac{8\pi a}{n}\cdot\frac{\sin\dfrac{\pi}{2}\sin\dfrac{n-1}{2}\dfrac{\pi}{n}}{\sin\dfrac{\pi}{2n}}$$

$$=\frac{8\pi a}{n}\,\frac{\sin\left(1-\dfrac{1}{n}\right)\dfrac{\pi}{2}}{\sin\dfrac{\pi}{2n}}$$

$$=8\pi a\cdot\cos\frac{\pi}{2n}\cdot\frac{\pi/2n}{\sin(\pi/2n)}\cdot\frac{2}{\pi}\,.$$

If we let the number of sides of the polygons increase, L_n approaches the arc length of the cardioid:

$$\operatorname*{Limit}_{n\to\infty}L_n=16a.$$

11. *Area.* The area enclosed by the circular arcs of Figure 9 is evidently double the area of the polygon plus the sum A_n of the $n-1$ circular sectors. Thus

$$A_n=\frac{1}{2}\,\frac{4\pi}{n}\sum_{k=1}^{n-1}r_k^2=\frac{4\pi a^2}{n}\sum_{k=1}^{n-1}2\sin^2 k\cdot\frac{\pi}{n},$$

or, since

$$2\sin^2 k\cdot\frac{\pi}{n}\equiv1-\cos k\cdot\frac{2\pi}{n},$$

$$A_n=\frac{4\pi a^2}{n}\left[n-1-\sum_{k=1}^{n-1}\cos k\cdot\frac{2\pi}{n}\right].$$

Using the identity

$$\sum_{k=1}^{n-1}\cos k\theta\equiv\frac{\cos\dfrac{n}{2}\theta\sin\dfrac{n-1}{2}\theta}{\sin\dfrac{\theta}{2}},$$

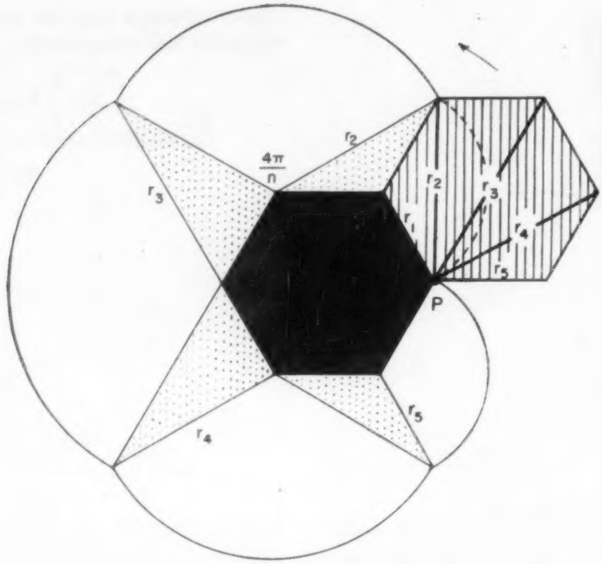


Figure 9

we have

$$A_n = \frac{4\pi a^2}{n}\left[n-1-\frac{\cos\pi \sin\left(1-\frac{1}{n}\right)\pi}{\sin\frac{\pi}{n}}\right]$$

$$=4\pi a^2,$$

a rather surprising result, since it is independent of n.

As n increases without bound, the sum of the areas of the polygons approaches $2\pi a^2$, and the area of the cardioid is thus $6\pi a^2$.

12. The following additional facts concerning the cardioid may be discovered by other means:

a. it is the *inverse* of the parabola with respect to its focus;
b. its evolute is another cardioid;
c. if s represents arc length from O to P, Figure 1, then

$$SR+RP+s=8a.$$

"A circular number when it is multiplied by itself begins with itself and ends with itself. For example $5\times5=25$. A spherical number is that which being multiplied by a circular number ends with itself; for example, $5\times5=25$ and this circle being multiplied by itself makes a sphere that is $5\times25=125$."—*Taken from A* History of Education *by Luella Cole.*

Breakthroughs in mathematical thought

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The "breakthrough problem" in our school mathematics program is of national concern. As such, it becomes the concern of all teachers of mathematics.

CHANGES in the educational program of our country are difficult to make and they take an exceedingly long time to accomplish. The present program of mathematics education, despite calls for reform, is for all practical purposes more than a hundred years old. Very few new ideas have been added since 1900 A.D., and there has been no real shift in direction. This is equally true of the undergraduate collegiate program in mathematics. The time is ripe for a breakthrough to a more modern and vitalized program. We have the support and encouragement not only of mathematicians, but also of fellow scientists, industry, and the public. The big question is, "Do we have the know-how and drive to accomplish it?" I think so.

Social breakthroughs are difficult to make, but so are those in the field of pure science. Initially, great discoveries have usually been met with rejection by those entrenched and practicing in the field. It sometimes takes a century of pleading, pushing, and re-education before a great idea is accepted. This is particularly true of great mathematical breakthroughs, which usually took centuries to permeate the culture of a society. Let us examine only a few of these great discoveries in the field of mathematics.

Perhaps man's first great triumph in mathematics was the invention of number. It is natural for anyone to attempt to measure how much he has of any material goods. When finally some man devised a scheme for giving a name to that property of a set of physical things that was totally independent of any qualities or characteristics of the things except their numerosity, their manyness, or the size of the set, the great breakthrough had been made. Now this same name could be applied to all equivalent sets, regardless of the make-up of the elements. Using these names of sets, it was easy to devise a set of counting numbers: zero, one, two, etc. . . . How many centuries it took for this system to be translated to the set of symbols that made the arithmetic we know today mere child's play, is a record of history.

Was this delay due to a lack of intelligence on the part of our ancestors? I doubt it. It was due to a societal structure that prevented communication of ideas, except to a selected elite, and that kept the masses in complete ignorance. Actually, it is only in the very recent history of Western civilization that mass education has been accepted, and it has proved a powerful asset for human living.

Man looks about his world—its skies, fields, rivers, oceans—and he tries to explain how it is shaped and how it can be navigated. At first he concentrates on his own environment, later on the whole earth, finally on the outer reaches of space. His explanation is usually in terms of surfaces, direction, distances, and the like. It took a long time before Euclid organized all the past experience and thinking of the human race into an idealized explanation of those phenomena, called geometry. Though concerned with describing their universe, men of his time were equally interested in investigating how we think

and arrive at correct deductions. Thus Euclid's geometry was a model not only of the universe but of the way men think logically. This tie-up of deduction with geometry prevented any systematic axiomatic treatment of other areas of mathematics for nearly 2,000 years.

Of course, one cannot long use number without concerning himself with the general relations that exist between numbers, and then algebra is born. Algebra itself afforded many breakthroughs in mathematical thought, and it is still doing so today. But the one great breakthrough was that which united quantity and space by means of a number-and-variable approach to geometry. It was Descartes who did this through co-ordinate geometry. Graphing, of course, had been known many centuries before Descartes published his paper on "The Method." But graphing or drawing pictures of curves was not the essence of "The Method": the important fact was that numbers and variables, or algebra, could be used to explain the relations of points, lines, and surfaces of space. In a sense co-ordinate geometry is algebraic geometry, and now, 300 years later, this idea is breaking through in full force. More and more, geometry is being studied by use of numbers and not synthetically.

In far less detail I mention a few more significant conquests of ideas. The "infinite" in mathematics was always an elusive and feared idea. From Zeno to Archimedes, to the mathematicians of the late seventeenth century, the limiting process and infinite series were presented as baffling works of the devil. Newton and Leibnitz, through the study of changing physical quantities, finally developed the differential and integral calculus and thus harnessed infinity, at least the "potential" infinity. However, one need only read Bishop Berkeley's tract in which he refers to derivatives as "the ghost of departed quantities" to see how difficult it was for the idea of a derivative to break through to the mathematical world. If it gained much quicker acceptance than other new ideas, it was doubtless because of its power in producing mathematical results.

Next came the severe criticism of the logicians that began with Saccheri, who was followed by Bolyai, Gauss, and Lobachevsky. The resulting breakthrough caused a cataclysm in mathematics that is still raging today. By pure thought, and an axiom that challenged the parallel axiom of Euclid, new geometries were established, perfectly consistent, unassailable in structure. Thus were new models made available to explain large and small universes that could be nothing more than fictions of the mind. At first rejected (Gauss was afraid to announce his results for this reason), within a half century these new ideas were leading mathematicians to one of the greatest heydays in mathematical history: that of the axiomatic structure of all mathematics. The works of Frege, Peano, Whitehead, of Russell, Hilbert, Veblen, Huntington, and of the mathematical or symbolic logicians of today are in large measure the result of Riemann's confirmation of the non-Euclidean and other geometries as respectable mathematical disciplines.

Looking for a unifying concept of an ever-growing body of knowledge, the mathematicians first grasped at number, then (especially the Arabs) at the geometrizing of algebra, then at the use of function. Today the theory of sets emerges as the most important of all these unifying ideas. It was Georg Cantor who had the courage and conviction to announce his theory of sets as a necessary reform for the foundations of mathematics. He, too, was assailed bitterly by mathematicians of stature, including Brouwer and Kronecker, as presenting unacceptable nonsense. Eighty years later, however, we find the theory of sets becoming the basis of structure in algebra, geometry, and analysis and the most clarifying idea to enter the study of mathematics throughout its long and continuous history.

The whole story of the growth of mathematics is one that depicts a striving to be

rid of the concrete—to be pure and abstract. If mathematical tools were created to help man solve particular practical problems, mathematicians developed out of these tools a structure of knowledge, with pattern or forms, that became a way of thinking about abstract entities. There has been a continuous stair-climbing, from learning mathematical skills or tricks to the development of concepts and ideas that give intelligent understanding to the skills and tricks. Today mathematics has reached a new height in its abstractions, its structure, and its power of penetration into pure thought. Mathematics, on the frontiers of research, is literally out of this world.

Up to 1900, as Warren Weaver has shown in his article "Science and Complexity,"[1] mathematics enabled us to solve problems of "simplicity," that is, those with only one or two variables, and usually expressible in differential equations. By 1900, methods of probability and statistics were perfected that enabled us to solve problems of "disorganized complexity," i.e., problems with millions of variables subject to laws of chance. Today mathematics, through the development of electronic computers and a variety of new mathematical subjects, is showing the way to the solution of problems of "organized complexity," that is, problems with many variables operating simultaneously.

The advance of mathematical research and the stagnation of high school and college courses in mathematics have caused a huge gap between what we teach and the way research mathematicians conceive of their subject. Suddenly the mathematicians have realized this alarming state of affairs and have begun to show a tremendous interest in the high school curriculum. They have looked at our textbooks and curriculum and express deep concern. It is not the methods by which we are teaching in the elementary and secondary schools that concern them, for they recognize that teachers at this level, as teachers, are quite as good and usually superior to college and graduate school teachers. It is *what* we are teaching that concerns the mathematicians. This is good. The mathematicians desire to close the gap between 19th-century high school mathematics and 20th-century mathematical thinking at research frontiers.

This state of affairs is true not only of mathematics, but of most of the sciences. In physics it was so bad that a group of physicists, with headquarters at the Massachusetts Institute of Technology, and with Professor Zacharias as leader, have produced three volumes designed as textbooks for a modern physics course in the high school. These texts are pure science and devoid of all technology. They were written by college professors because they had no confidence in the ability of high school teachers or the authors of present textbooks to manage such material. But because the high school teacher was bypassed, because pure physicists were totally ignorant of the high school youngsters' capacity for learning and of how to teach them, the books are proving difficult and applicable only for students in the upper 10 or 5 per cent intelligence bracket. This program will meet much resistance. The breakthrough to a better physics program may be delayed.

Pure mathematicians have been warned against the mistake of bypassing the high school teachers. High school teachers will teach what they know and what they feel appropriate and not what a group, no matter how high its standing, dictates. Fortunately for us, there are a number of outstanding mathematicians who know this and who are ready to work with the teachers in producing a truly modern, teachable program. It is now the task of the teacher to meet this opportunity, and with this problem I come to the last phase of my remarks.

We are now on the verge of a breakthrough in our one-hundred-year-old high school curriculum. There are many salient

[1] Warren Weaver, "Science and Complexity," *American Scientist*, XXXVI, No. 4 (1949).

points along the front at which gains are being made and new attacks readied. I shall mention only a few. The University of Illinois Committee on School Mathematics is engaged in a study that has as its basic purpose the development of a mathematics curriculum that is modern in spirit, correct in its presentation, and as rigorous and pure as the maturity of high school intellect permits. To understand this program takes, on the part of the teacher, considerable mathematical maturity and a rather comprehensive knowledge of recent developments in mathematical thought. The Commission on Mathematics of the College Entrance Examination Board, consisting of high school teachers, educators, and mathematicians, has developed a program that permits adaptation from the present program through elimination of unnecessary obsolete material, a reformulation of the algebra, geometry, and trigonometry, and an addition of a very few modern mathematical concepts that give unity and structure to the entire program. The understanding of this program for proper teaching likewise demands more knowledge of mathematics than most teachers of high school mathematics have had, but not so much that it cannot be obtained by in-service or summer study by the teachers. At Yale, under the guidance of Professor Begle, a group of mathematicians and high school teachers met to study and write textual materials for the secondary school that are modern in flavor and content. Such a project will demand great thought and considerable mathematical understanding on the part of the high school teachers involved. All over the country various groups, large and small. are embarking on projects to produce a better mathematics program. Will they succeed?

The answer lies in the minds of the classroom teachers of mathematics. Unless a teacher is a scholar in his field, one who knows and can speak with authority, he will be unable to judge the merit and feasibility of the programs mentioned before, or of any other new program. No teacher has the right to condemn or to praise a program which he is in no position to evaluate honestly. The good mathematics teacher must become a respected scholar. To do this he must keep up to date by reading, study, attendance at professional meetings, and taking in-service courses in new developments in mathematics, its applications, and its teaching. No teacher can do this when daily he has five classes to teach, a study hall to supervise, and extracurricular duties, including perhaps a P.T.A. meeting to attend, plus family responsibilities, some of the last rather heavy. The teacher must demand removal of excessive teaching assignments and administrative and nonteaching duties; he must demand released time and travel funds for attendance at professional meetings; he must demand an adequate salary to live comfortably.

A teacher should be abreast of all that is going on in his field; he should afford, develop, and use a professional library of his own (not the school's); he should give unselfishly of his time and mind to this profession. He should teach far beyond the pages of a textbook, because he is capable and because he has enthusiasm for his subject; he should excite interest and a desire to learn in his students, because he himself is a master and lover of mathematics; he should encourage students to go on with their study of mathematics by the use of their own wits, because he is doing the same thing. Such a teacher will be respected and honored by his students, even more so by the citizens whom he serves, and most of all by his colleagues who work at the collegiate and graduate levels.

With scholarly members the National Council of Teachers of Mathematics can and must become the spokesman for the mathematics education of the youth of America throughout the elementary and secondary schools. America should need no group of mathematicians, no matter

how distinguished, to dictate its elementary mathematics program: the Council should have these scholars within its own ranks. The National Council of Teachers of Mathematics encourages all mathematicians to be active members of its organization, so that there may not arise again the gap that now exists between school and modern mathematics. The Council must be the voice of the mathematics teachers of America. It must speak out boldly and responsibly. The Council must at all times co-operate with the other learned societies both in mathematics and in education, but it must always remain independent. It must, through its elected leaders, its membership, and its journals and publications, attain public and professional respect and confidence that will give to it the money and encouragement for continued growth. We can and must reach the place where the public will come directly to the Council when it desires authoritative advice on mathematics education at the elementary and secondary school levels. The Council must speak for the education of all American youth and not only for the college-bound girl and boy.

Some say this is not possible. Some say it is silly to think of high school teachers as scholars, as people who can get larger salaries, deep respect, and become devoted to their professions. I say that it is possible, and now is the time for the breakthrough. To all members of the National Council of Teachers of Mathematics, to all mathematics teachers, I say forsake complacency, be brave, and go boldly forward. Diligence, persistence, study, and faith in yourself and in your subject will show you the way.

The Mathematics Student Journal

The solutions below are to problems appearing in *The Mathematics Student Journal*, January 1959.

Cat and Mouse. The game usually leads to a position something like this: the Cat is at 8, the Mouse at 13, and it is the Cat's turn to move. If the Cat could abandon its move and stay in 8, it would clearly have won; but it is compelled to move. The Cat's problem is essentially to lose a move. It can do this by passing through position 1. The Cat, then, should ignore the Mouse completely at first. It should go straight up to position 2, then to 1, then to 3. It now has no difficulty in catching the Mouse.

Problem 128. The problem is essentially this. O is the centre of a circle. BA and BC are perpendicular tangents to the circle, touching the circle at A and C. OB meets the circle in D. The length BD equals a and is known. Deduce the diameter of the circle.

Clearly $OABC$ is a square, of side r, equal to the radius of the circle. Hence $OB=r\sqrt{2}$. $a=OB-OD=r(\sqrt{2}-1)$. Multiply both sides by $\sqrt{2}+1$. Thus $r=a(\sqrt{2}+1)=2.414\ a\cdot d=2r$ gives the desired result.

Problem 129. In Alan Preston's first table, the n^{th} number in the row for a is $2n+1$. The numbers for b are obtained by summing an arithmetical progression. The usual formula $S=\frac{1}{2}n\{2a+(n-1)d\}$, for summing the series with initial term a and difference d, gives $b=2n^2+2n$. Thus $a^2+b^2=(2n+1)^2+(2n^2+2n)^2=4n^4+8n^3+8n^2+4n+1=(2n^2+2n+1)^2$. Thus $c=2n^2+2n+1$, which is a whole number.

Similar reasoning for his second table shows $a=4n+4$, $b=4n^2+8n+3$. Some shapeless computations may be avoided by noticing that $c=b+2$. Hence $c^2-b^2=(c+b)(c-b)=2(c+b)=2(8n^2+16n+8)=16(n^2+2n+1)=a^2$. We thus verify that the whole number sequences specified by the formulas do give right-angled triangles.

Tricks and Why They Work. The Test of Accuracy at the end is algebraically the following:

Think of a number.	X
Add 37.	$X+37$
Multiply by 500.	$500X+18,500$
Add 64.	$500X+18,564$
Multiply by 1,000.	$500,000X+18,564,000$
Add 859.	$500,000X+18,564,859$
Double.	$1,000,000X+37,129,718$
Add 2,870,282.	$1,000,000X+40,000,000$
Divide by a million.	$X+40$
Take away the number first thought of.	40.

The trick thus works for all numbers. It can of course be used as a purely arithmetical exercise by starting with a particular number.

The algebra of sets in the teaching of trigonometry

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While the use of the algebra of sets suggested in this article is not extensive, it does serve to make use of ideas that are presently being proposed by some groups for high school classes.

THE RECENT LITERATURE of mathematics education contains many references to new emphases and points of view in the presentation of subject matter as well as some indication of new subject matter that is sufficiently important to warrant its possible inclusion in the secondary school curriculum. The concluding chapter of the most recent yearbook of the National Council of Teachers of Mathematics is devoted to this problem of the changing curriculum. It also includes a list of several of the groups which are active in organizing and experimenting with new programs [6].* Some more specific references are listed in a recent issue of the National Education Association publication *Review of Educational Research* [4].

It would seem that the classroom teacher who is not a participant in one of the experimental programs can best do his part by using new approaches in the presentation of traditional subject matter, especially when the newer methods seem to aid in pupil understanding. It is the purpose of this paper to indicate one new approach in which the algebra of sets is introduced in teaching a topic in trigonometry.

The basic approach presented here first occurred to the writer about two years ago while he was teaching that unit in trigonometry in which trigonometric functions are defined in terms of co-ordinates. He has found it to be a successful approach that is interesting to the students.

In teaching trigonometry it is necessary to place some emphasis on developing an understanding and knowledge of the signs of the functions in the various quadrants, since the students need to have this information at their command in the work that follows. Hence the students are asked such questions as "What is the s-i-g-n of the s-i-n-e in the third quadrant?" and, eventually, "In which quadrant is θ if cos θ is negative and sin θ is positive?" The emphasis is always on the signs of x and y in the four quadrants. Accordingly, in the discusssion below, "sine" may be replaced by "ordinate," "cosine" by "abscissa," S by Y, and C by X, and the lesson may be presented at the ninth-grade level in connection with graphing. The writer prefers to carry out the development in such a manner that the students supply many of the results, which will be given here in the form of a synopsis of the teacher's presentation. (Some parenthetical remarks on notation are addressed to the reader.) This development is usually spread over at least three class periods, along with some of the standard presentation of trigonometry.

* Numbers within brackets refer to the References at the end of the article.

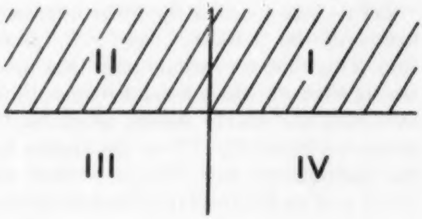
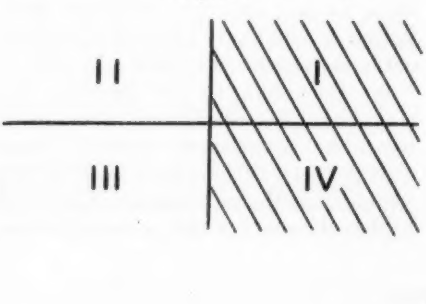


Figure 1

The sine is positive for angles in standard position with terminal side in quadrant I or II (usually stated briefly as "The sine is positive in quadrants I and II"). We recall that a set can be interpreted as a collection—for example, the set of all pieces of chalk on the chalk rail or the set of all students in the class. Suppose we write S to represent the set of quadrants in which the sine is positive. We may write $S=\{I, II\}$ to indicate clearly the membership of the set S. Graphically, S may be represented by the shaded area in Figure 1. Similarly, let C represent the set of quadrants in which the cosine is positive. Then $C=\{I, IV\}$ and is indicated graphically by the shaded area in Figure 2.

Sometimes we wish to discuss the quadrants in which the functions are negative. It seems convenient to let $\overline{S}$ represent the unshaded area of Figure 1, so that $\overline{S}=\{III, IV\}$. We may at first think of the bar over the S in terms of the minus sign, although a technical name for $\overline{S}$ is the *complement* of S. (Some authors use S' or $\sim S$ for the complement of S.) Similarly, the unshaded portion of Figure 2 represents $\overline{C}=\{II, III\}$.

Figure 2



Now we may use our language of sets to answer the question, "In which quadrants may an angle terminate if its sine and cosine are both positive?" Any such quadrant has to be included in both S and C. Therefore quadrant I is the only answer. In the notation of sets this is $\{I\}$, called the *intersection* (or logical product or simply the product) of S and C. It is written as $S\cap C$ (alternate notations are $S\cdot C$ or simply SC), and is sometimes read "S cap C." Graphically, we have the region that is shaded in both Figures 1 and 2, as shown in Figure 3 by the region which is shaded in two ways.

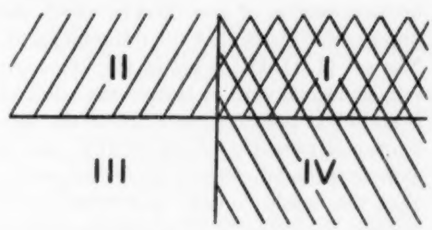


Figure 3

Since $S\cap C$ is included in S and also in C, it is said to be a *subset* of each. Similarly, $S\cap\overline{C}=\{II\}$, $\overline{S}\cap C=\{IV\}$, and $\overline{S}\cap\overline{C}=\{III\}$ are answers to questions about sign combinations for the trigonometric functions. We notice also that $S\cap S=S$, $C\cap C=C$, etc.

It is only natural to ask about $S\cap\overline{S}$ and $C\cap\overline{C}$. Because there is no quadrant in which a given trigonometric function has two signs, we have nothing to write within the braces, and so we write $\{\ \}$ for $S\cap\overline{S}$ and for $C\cap\overline{C}$. This is called the *null set* or empty set. Another way of indicating the null set more compactly is by using the symbol $\varnothing$. (Some writers use Z or O for the null set.) Hence $S\cap\overline{S}=\varnothing$, $C\cap\overline{C}=\varnothing$; and we notice that $S\cap\varnothing=\varnothing$, $C\cap\varnothing=\varnothing$, etc.

The language we have been using is basic in what is called "Boolean algebra" after the Englishman George Boole (1815–

1864). One of the problems Boole considered in a famous book published in 1854 [1] concerns the meaning of a question such as, "In which quadrants may an angle terminate if either its sine or its cosine is positive?" He pointed out that there are two interpretations of such a question. In our case these two interpretations correspond to including or excluding the quadrant in which both functions are positive. In other words, we may include all of the shaded area in Figure 3 or we may include only the portion which is shaded once.

There seem to be more useful applications of the former approach, however, in which we have $\{I, II, IV\}$ as the answer in the notation of sets (and in which the order of the elements has no significance). The set $\{I, II, IV\}$ is called the *union* (or join or logical sum or simply the sum) of S and C. It is written as $S \cup C$ in the conventional notation of set theory, and is sometimes read "S cup C." (We notice that the words "union" and "cup" both contain the letter "u.") It is apparent that $S \cup C$ contains both S and C as subsets, and that it is the set of quadrants in which at least one of the two functions, sine and cosine, is positive.

Similarly, $S \cup \overline{C} = \{I, II, III\}$ consists of those quadrants in which the sine is positive and/or the cosine is negative; and we also have $\overline{S} \cup C = \{I, III, IV\}$ and $\overline{S} \cup \overline{C} = \{II, III, IV\}$, as well as $S \cup S = S$, $C \cup C = C$, etc. Once more, it seems natural to ask about $S \cup \overline{S}$ and $C \cup \overline{C}$; but since a given trigonometric function is either positive or negative in every quadrant, each of these two sets is $\{I, II, III, IV\}$ and consists of all the quadrants. The set $\{I, II, III, IV\}$ is called the *universal set* of our discussion, which is consistent with the fact that it contains everything that is being discussed. Another way of indicating it is by the symbol 1. (Some writers use U for the universal set.) Hence $S \cup \overline{S} = 1$, $C \cup \overline{C} = 1$; and we notice that $S \cup 1 = 1$, $C \cup 1 = 1$, etc. We also readily see that $S \cap 1 = S$, $C \cap 1 = C$, etc.

Let us now consider the other interpretation of the "either . . . or . . . " question. This time we include only that portion of Figure 3 which is shaded once, thus excluding the doubly shaded quadrant I; hence we have $\{II, IV\}$ as the answer in the notation of sets. This is written as $S+C$, and we shall call it by its descriptive name, the ***separated sum*** of S and C. (In a rigorous development it would be called the "ring sum.") We note that $S+C$ is the set of quadrants in which only one of the two functions, sine and cosine, is positive. Furthermore, $S+\overline{C}$ consists of those quadrants in which either the sine is positive or the cosine is negative but not both of these are true; and it is $\{I, III\}$. Similarly, $\overline{S}+C = \{I, III\}$ and $\overline{S}+\overline{C} = \{II, IV\}$.

Now, $S+S$ means all quadrants included in one but not both of the named sets; but since the named sets are both S, there are no such quadrants.

Therefore $S+S=\varnothing$, and, similarly, $C+C=\varnothing$, $\overline{S}+\overline{S}=\varnothing$, etc. These separated sums are quite different from the results we had above for the unions of the same sets; but $S+\overline{S}=1$, $C+\overline{C}=1$, etc., just as for the unions. The reason is clear when we recall that S and $\overline{S}$ have no common element and hence are already separated.

Incidentally, Boole never used the notation $S+C$, always indicating this set by what we would write as $(S \cap \overline{C}) \cup (\overline{S} \cap C)$. It is easy to check that this correctly represents $\{II, IV\}$. Likewise, instead of $S \cup C$ he used the equivalent of $(S \cap \overline{C}) \cup (S \cap C) \cup (\overline{S} \cap C)$. Checking this, we find it equal to $S \cup (\overline{S} \cap C)$ and to $C \cup (\overline{C} \cap S)$, all three expressions representing $\{I, II, IV\}$. It is also easily checked that $S \cup C = (S+C) \cup (S \cap C)$. As a final result on separated sums, we note that $S+1=\overline{S}=\{III, IV\}$; the set of quadrants included in one but not in both of the sets S and 1.

In conclusion, let us illustrate two theorems of Boolean algebra. By $\overline{S \cup C}$ we mean the complement of $S \cup C$. Since $S \cup C = \{I, II, IV\}$, we have $\overline{S \cup C} = \{III\}$ representing the only quadrant in which neither the sine nor the cosine is positive,

as shown by the unshaded region in Figure 3. But this same result may be obtained in another way as the quadrant in which both functions are negative, that is, by using $\overline{S}\cap\overline{C}$. Hence we have verified that $\overline{S\cup C}=\overline{S}\cap\overline{C}$, which is a general theorem in Boolean algebra. It is equally simple to verify that $\overline{S\cap C}=\{\text{II, III, IV}\}=\overline{S}\cup\overline{C}$. Additional results may be obtained by introducing $T=\{\text{I, III}\}$, the set of quadrants in which the tangent is positive, as well as the corresponding sets for the other trigonometric functions.

There are other results of Boolean algebra that may be illustrated with the model presented above. An excellent reference that gives further examples of the intuitive introduction of the algebra of sets is Chapter 1 of Part II of *Universal Mathematics* [8]. A more detailed presentation is given in a chapter of *Insights into Modern Mathematics* [5]. The presentation in *What Is Mathematics?* [2] uses a notation different from the one in this paper, inasmuch as the symbol $A+B$ is used in place of $A\cup B$ for union. Another excellent reference in which this notation is used is Chapter 23 of Freund's book [3], a presentation with which better high school students should have little difficulty. The last two references mentioned do not discuss the idea of separated sum, hence no confusion results from the notation. A more extensive presentation which includes both $A\cup B$ and $A+B$ (called "ring sum") is that by Stabler [7].

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Letter to the editor

Dear Sir:

After reading the very interesting article, "Things and Un-Things," by W. W. Sawyer, which appeared in the January 1958 issue of The Mathematics Teacher, I thought it might be of interest to present another approach to the problem of explaining multiplication of negative numbers.

It seems to me that the real difficulty in explaining multiplication of negative numbers lies in the commonly accepted definition of multiplication. The term usually refers to a process equivalent to repeated addition. However, if we think of multiplication as repeated addition when the multiplier is positive and as repeated subtraction when the multiplier is a negative number, one can easily show that repeated subtraction of a negative number will produce a positive result.

A restatement of the definition of the term, multiplication, might read as follows: "Multiplication is the process of finding a quantity (the *product*) resulting from the repeated addition or subtraction of a given quantity (the *multiplicand*) to or from zero, taken as many times as there are units in another quantity (the *multiplier*); addition being employed when the multiplier is positive and subtraction being employed when the multiplier is negative."

Sincerely yours,
Ned Harrell,
2140 Elizabeth St.,
San Carlos, California

Editorial Note: While many teachers use the ideas suggested by Mr. Harrell and Mr. Sawyer, it does seem that recent efforts are away from this approach to one reflecting a sounder mathematical framework.

Main issues concerning the Soviet scientific, engineering, and educational challenge[1]

NICHOLAS DEWITT, *Russian Research Center, Harvard University, Cambridge, Massachusetts.*
The author of Soviet Professional Manpower—Its Education, Training and Supply *sets forth issues facing America—its citizenry and its teachers.*

This is merely a guide to and a digest summary of the main issues in the by now extended controversy about the so-called "challenge" of Soviet education and science. The chief intent of this summary is not so much to provide answers, but mainly to raise and to focus attention on what seem to have emerged as the main issues.

Issue 1: Public understanding, and the government's and academic community's responsibility in the assessment of Soviet scientific and educational capabilities

We look at Soviet Russia from both sides of the binoculars at the same time. We keep twirling the binoculars at will. One day we make the Russians omnipotent monsters, and the next day we claim they are midgets incapable of real achievement in the sciences or in education, as well as in other fields. The public is misled into a false sense of security in periods of "negative" appraisal, and driven to hysteria in the phases of "positive" reappraisal of Soviet capabilities. As our judgment shifts, this in turn plunges us into policies and practices of a sporadic, short-lived nature. Particularly in science and education our sights should be set on long-run policies. In these policies we should be concerned with *our own* requirements and our own strength and not primarily with those of the Russians. If we keep adjusting our policies to our shifting judgment of Soviet capabilities, we will be bound to lose our leadership.

Issue 2: The long-run implications of education and science policies

Knowledge is the foundation of power. In this divided world, *not* the present stock of resources, *not* the current advantages in wealth and technology, *not* the present stock of arms and weapons, but the brains and minds of men will decide the destiny of man and society in the long run. It takes 15 to 17 years just to educate a man; it takes another ten to make him an experienced engineer, an efficient researcher, or an experienced medical doctor. The Russians recognized this fact three decades ago, and the things that we witness today are not the end-product of today's effort. They are the result of a persistent drive that started long ago.

Within this context of long-run policies is the Soviet claim that if not today or tomorrow, then in a decade or a generation or two hence, the absolute technological, economic, and military supremacy of communism and the Soviet Union will triumph. Their leaders think that their educational

[1] A statement prepared for Yale Conference of President's Committee on Scientists and Engineers.

program is aiding them in this goal. And this orientation justifies their outlook on education as being functional in nature. While we often say that the all-sided development of the individual is our primary concern in education, the communists do not believe in education for education's sake. Their main objective is to offer functional education so as to train, to mold, to develop the skills, the professions, and the specialists required by their long-run development programs. These specialists must be capable of performing to the best of their ability the tasks of running the increasingly complex industrial and bureaucratic machinery of the communist state. And in order to accomplish this task, the Russians were, they are, and they will be training an army of scientists and technologists.

Issue 3: Citizen responsibility and the problem of leadership

"To lead or to be led"—this is the main question. In the Soviet Union the central government and its local intermediaries instigate and promote action, as well as channel and control the execution of specific policies. Whatever we may think about the Soviet manpower problem, educational policies, or scientific effort, we have to keep in mind the fact that we are dealing with a *centrally* controlled effort. By way of contrast, in a pluralistic society like ours, we say that education is everybody's business. A multitude of local educational authorities set standards and control over educational practices. Our scientific efforts, save for those phases related to national defense, are un-coordinated. Under such conditions we cannot proceed with decisive action without *informed* public opinion and public support. Clearly, we do not want to change this. But citizen groups, institutions, localities, and states can orient their action only if there *are* national policies in the field of human resources, education, and science and if there is strong national leadership to prompt them to such action.

Issue 4: Financial support of education

Education is the main determinant of trained manpower, and it is a society's investment in education that decides the *flow* and preservation of manpower as a national resource. The gross national product of the Soviet Union is about *two-fifths* of ours, but their annual investment in education is about the same as ours. We spend about 3 to 3.5 per cent of the American gross national product on education, while on a relative scale the Soviet Union spends at least twice as much, or some 7.5 to 8 per cent of the Soviet GNP.

Issue 5: Psychological motivations and material incentives

In the Soviet case, the moving ideological orthodoxy is Marxism, which claims to be a science. Communist propaganda extols the objective, cognitive powers of science. It acclaims the supremacy of science in productive processes and the organization of society. In short, Science with a capital S is on permanent display in the dogma. To this is added the traditional Russian intellectual attitude toward knowledge as being omnipotent and "good," no matter for what; and we get almost a mania complex on a national scale for science and knowledge. This is accompanied by material incentives. The rewards of the intellectual elite are generous, indeed very generous, compared with the drabness, shabbiness, and deprivations of the average Soviet worker or peasant. It is quite common for engineering salaries to be *three* times higher, professional salaries to be *five* times higher, research scientists' salaries to be *seven* times higher, and the salaries of outstanding designers and academicians to be *ten* and more times higher than those of the average wage earner. Prizes, honors, medals, scarce new apartments, suburban homes, private cars, and scores of intangible favors and privileges—all are bestowed by the Soviet state upon those who show that they are good performers and producers of products, of research, of ideas which can directly or

indirectly enhance industrial and military power or score a political or propaganda advantage.

For our part, all this suggests how much a scientist, an engineer, a technician or professional is worth, or should be, and how much in the way of prestige and incentives we should be willing to grant them in our society.

Issue 6: The issue of numbers

As of 1956, the Soviet Union had an estimated 20 million persons with completed secondary nonspecialized education. We had probably twice as many. At the beginning of last year, 1957, the Soviet Union employed in its civilian labor force (excluding the military, that is) some 2,630,000 professional specialists with completed *higher* education, and about 3,600,000 subprofessionals and technicians with specialized secondary education. We have six million college graduates. Among the higher education graduates employed in the Soviet economy there were about 720,000 engineers, versus 600,000 in the U.S.A.; 160,000 agricultural field specialists, about the same as we have; 330,000 medical doctors, which is 100,000 more than we have; and some 500,000 science and mathematics teachers and university-trained scientists in nonapplied fields, which is one third more than we have. Altogether, there were about 1,700,000 people in scientific and engineering fields in the Soviet Union, which represents almost *two thirds* of *all* Soviet higher education graduates. We had 1,370,000 in science and engineering fields, which is just about *one fifth* of our college graduates. Thus, although we had better than *twice* as many college graduates, in the respective categories of science and engineering in the past year the United States already employed a smaller number than did the Soviet Union.

Now to complete this numerical picture, we have to consider the current flow, the new additions to this stock. This year the Soviet general education 10-year schools graduated about 1,800,000 youths, and of these about 40 per cent continued their education further. Subprofessional specialized schools graduated about 500,000, among whom almost 180,000 were technicians and engineering aides for Soviet industry. Institutions of higher learning and universities graduated about 280,000, of whom 70,000 were professional engineers and about 80,000 were in other scientific fields (biological, medical, and agricultural fields, science teachers and university-trained researchers). While the totals of higher education (college) graduates were roughly comparable in the two countries, in scientific and engineering fields Soviet graduates outnumbered our annual additions better than *two to one*. Such a Soviet lead has been in evidence for the last six years, and there is no sign that it will subside in the near future. This year enrollment in Soviet engineering schools edged up to 700,000. Thus, to sum up, the present stock of trained men in science and engineering for the two countries, the United States and Russia, slightly favors the Russians already, and in addition they have, and will continue to enjoy, a better than *two-to-one* edge in the new flow of graduates.

Issue 7: The issue of quality

By and large, Soviet higher education graduates are certainly not any worse prepared than our own. Some are willing to go one step further and say that in the engineering and science fields the training of Soviet graduates is probably of *better* quality than that of our own graduates in these fields. The standard curriculum, the strong subject matter instruction, and the selective processes in Soviet secondary education resemble those of European-type education and account for high qualitative standards on the secondary-school level. Almost half of all instruction time is spent on the sciences and on mathematics in Soviet secondary schools. It is the secondary schooling in the Soviet case that accounts for early and vigorous ex-

posure to the sciences, and makes for better student material from which to choose for further training. The 5- to 6-year training in higher educational institutions is fully adequate not only for providing the student with basic fundamentals and engineering tools, but also for developing professional specialization.

Issue 8: The issue of the internal implications of the Soviet scientific effort

Soviet science indeed does serve the state, its military, political, and power objectives. But in serving these objectives it often fulfills scientific purposiveness and thus gets full benefit and full credit. And this is what we often fail to grasp in trying to understand the Soviet effort. We repeatedly and persistently underrate the Soviet potential in science and technology. We forget that the Russians have reached such a position with their scientific army that they can achieve any research or engineering objective that we can, but they cannot achieve all of them at once. And the scale of priorities dictated by the planners is very much in evidence. At times these priorities and political dictates result in genuine scientific accomplishments and at others lead to obvious "scientific" nonsense. The army has in its ranks brilliant soldiers as well as outstanding political charlatans and scientific clowns. And too often we have made our judgment without looking at both sides of the coin.

Have you read?

BERNSTEIN, ALEX, and ROBERTS, MICHAEL. "Computer Versus Chess Player," *Scientific American*, June 1958, pp. 98–105.

If you have a boy in your class who is a chess player, by all means have him read this article. He needs to know how to play chess, but he does not need to know how to operate a computer. Why chess and the computer? Because each chess game is new; there are at least 10^{120} possible different games. This is a lot of games: at one million games per second, it would take 10^{108} years to play them all. Even with the most efficient machine as opponent, no two games should be the same!

Your student chess player will be interested in reading how such games are programed, how the machine "thinks," what the human player must do, and how the computer checks any illegal moves he makes. The article takes a particular situation and carries it through. It makes fascinating reading. I'd like to play against a computer myself sometime.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

BOELM, A. W. "The New Mathematics," *Fortune*, June 1958, pp. 141–145, 150–158. "The New Uses of the Abstract," *Fortune*, July 1958, pp. 124–127, 152–160.

These two articles should be required reading for all mathematics teachers and counselors. The author has told the story of the present state of mathematics briefly and well. As one reads these articles he can feel the throbbing pulse of a dynamic mathematics. In general, the U. S. mathematician excels his fellows in other lands. The U. S. is called home by 600 of the 3000 creative pure mathematicians in the whole world. What about these people? They are generally young, they like to do their own work, they excel in asking questions, they have a highly developed sense of aesthetics, they have a knack of posing good problems, they seem to have a strange ability to capitalize on their intuition, and they love to push the abstract far beyond physical reality. Mathematics is the only branch of learning in which all the major theories of 2000 years ago hold true even while the world is being flooded with new ideas.

There are applied mathematicians, but they also relish the abstract. These men have as their goal the finding of new mathematical approaches to the solution of problems. The fields of statistics, topology, game theory, and group theory have all aided the applied mathematicians. There is a great deal to be mastered in modern mathematics, but it is no more difficult than the mathematics of 30 years ago.

What does this mean for you and me? We must teach for a changing—a growing—mathematics. We must start sooner, be more efficient, and look toward the future. Great opportunities lie ahead.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

On the rapid sketching of plane parametric curves

DONALD GREENSPAN, *Purdue University, Lafayette, Indiana.*

Curve sketching is a valuable visual aid for studying the properties of functions via algebraic techniques.

I. Introduction

In a recent article in The Mathematics Teacher (April 1957), the writer motivated and described a rapid method for sketching polar curves. Unfortunately, or fortunately, he used the customary raconteur device of closing the paper by noting that an analogous technique existed for parametric curves but that that was "another story." Requests concerning that other story have resulted in this note.

The parametric method, that is, $x=x(t)$, $y=y(t)$, for representing a function is extremely useful. Complex graphs, such as cycloids, are easily represented in this form, while the problem of finding components of a tanget vector to the curve is usually easily resolved. Nevertheless, the "hammer and tongs" technique of plotting myriad points to draw the graph is exceedingly laborious, often fruitless, and rather unexciting. The method to be proposed will usually require only from five to 30 seconds to yield an informative sketch of the function. The difficulty of describing the procedure in words far exceeds in difficulty the procedure itself. Finally, it is important to indicate that the method below yields the graph directly, that is, without elimination of the parameter. That this is of importance is clearly displayed in the following two curves:

Curve 1. $x=\cos t$, $y=\cos^2 t$.

If the parameter is eliminated by noting that $x^2=\cos^2 t=y$, then the resulting curve is: $y=x^2$. But this curve contains, for example, the point (2,4). However, x cannot have the value 2 on the original curve since $-1\leq x\leq 1$. It follows then that elimination of the parameter has introduced extraneous points (and quite a few of them).

Curve 2. $x=t^5+t^3-t^2\sin t$,

$$y=t^{13}+7t^4-e^t\cos t+t^2\log t.$$

Here the process itself of eliminating the parameter is something of a mystery.

II. Method of sketching

In general terms the method is as follows. (The example below should be used in conjunction with this description.) Draw two co-ordinate systems, an X-T system, called system I, and a Y-T system, called system II, as displayed in Figure 1. It is essential that the scales of the T axes in both systems be the same.

Now draw: $x=x(t)$, in I, and $y=y(t)$, in II. Consider then the following ranges of t: $t=0$, $t>0$, $t<0$. For $t=0$, record from I and II the corresponding values of x and y. Suppose these are x_0 and y_0, respectively. Plot then the starting point (x_0, y_0) in the X-Y plane. Consider next t increasing from $t=0$, i.e., the values $t>0$, and note the resulting variations of x and y from I and II. Record these variations in the X-Y plane by a curve. Finally consider t decreasing from $t=0$ and note the resulting variations of x and y from I and II. Record these variations in the X-Y plane, and the graph is complete.

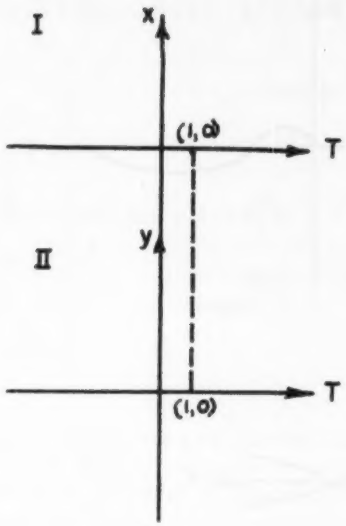


Figure 1

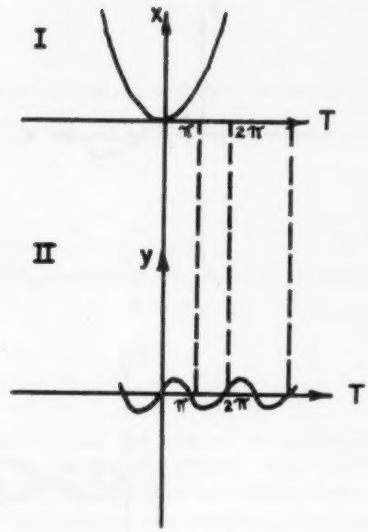


Figure 2

Example: $x=t^2$, $y=\sin t$

Steps:

(*a*) Sketch $x=t^2$, in I, $y=\sin t$, in II. (See Figure 2.)

(*b*) If $t=0$, then $x=0$, $y=0$. Hence, plot $(0, 0)$ in the X-Y plane. (See Figure 3.)

(*c*) As t varies from 0 to $\pi/2$, y varies from 0 to 1 and x varies from 0 to $\frac{\pi^2}{4}$. This is recorded in Figure 4.

(*d*) As t varies from $\pi/2$ to π, y decreases from 1 to 0 and x increases. It is important to note that this increase in x is larger than the previous one. Its exact size will not be used. The result is shown in Figure 5.

(*e*) As t varies from π to $3\pi/2$, y decreases from 0 to -1 and x increases. This increase in x is again larger than the immediately previous one. The result is displayed in Figure 6.

(*f*) As t varies from $3\pi/2$, to 2π, y increases from -1 to 0 and x increases. Again, the increase in x is larger than the immediately previous one. The result is in Figure 7.

The behavior of the graph for $t\geq 0$ is now apparent. Discussion of $0>t\geq -(\pi/2)$; $-(\pi/2)\geq t\geq -\pi$; $\cdots$, yields analogous statements to those described above, with the final graph extending infinitely to the right. (See Figure 8.)

Figure 3

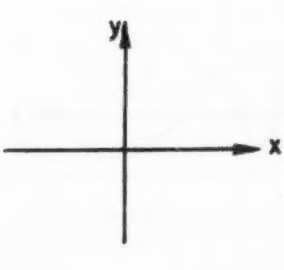


Figure 4

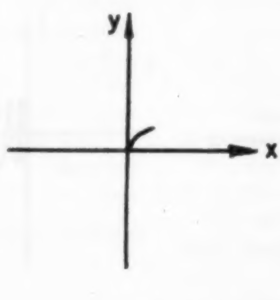
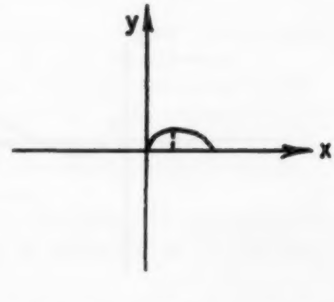


Figure 5



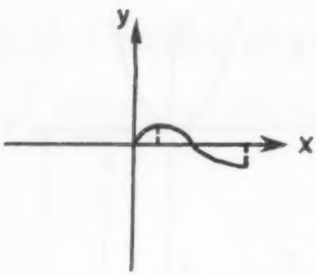


Figure 6

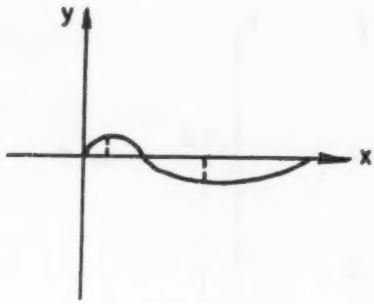


Figure 7

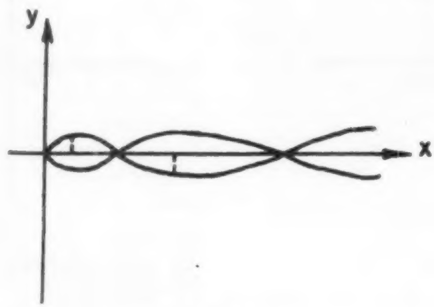


Figure 8

III. Concluding remarks

In practice, of course, only one X-Y graph is used and no discussion need be written.

The writer has found that this technique generalizes nicely to three dimensional curves and is superior in many respects to the representation of space curves as the intersection of two surfaces.

Other examples of plane parametric curves which are easy to sketch by the indicated method, but exceedingly difficult by point plotting, are demonstrated in Figures 9 and 10.

Finally, the writer has found the method to be an excellent aid in developing mathematical interest and initiative in beginning students. Because of the simplicity of the method and the ease with which one can write down a set of parametric equations, students are readily willing to create their own interesting parametric curves.

Figure 9

(*a*) $x=t^2, \quad y=\cos t$

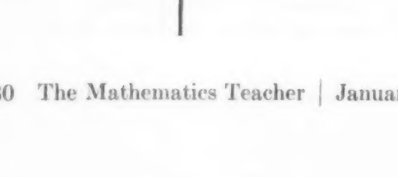
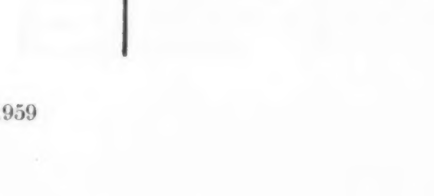


Figure 10

(*b*) $x=e^t, \quad y=\cos t$



Edited by Howard Eves, University of Maine, Orono, Maine

A historical puzzle

by N. A. Court, University of Oklahoma, Norman, Oklahoma

Do the altitudes of a tetrahedron meet in a point? It would be unwise to accept, on a bet, the challenge of answering this question by "yes" or "no": the answer would be incorrect in either case.

The proposition that the three altitudes of a triangle meet in a point is not found in Euclid's *Elements* (ca. 300 B.C.). But Archimedes proves this proposition in Lemma 5 of his *Book of Lemmas* [1].* He even notices that a triangle and its orthocenter (that is, the common point of the triangle's altitudes) form an orthocentric quadrilateral [2]. More than two thousand years later the latter property was rediscovered by the statesman and scholar L. N. M. Carnot (1753–1823) [3].

Did the existence of the orthocenter of a triangle suggest to the Greek geometers the idea of looking for an analogous point in connection with a tetrahedron? They knew the tetrahedron, which they preferred to call a triangular pyramid, and they considered an altitude of this solid in connection with the study of its volume [4]. They thus had all the elements necessary to "pop the question." Moreover, even before the time of Euclid, the Greeks were already familiar with the regular pyramid as one of the five regular solids of Plato (430–349 B.C.). And if the question had occurred to them, they could not have failed to find the answer to it, for that answer hinges on nothing more complicated than the following simple theorem.

THEOREM: *If two altitudes of a tetrahedron have a common point, the edge joining the two vertices involved is perpendicular to the opposite edge.*

Indeed, if the altitudes DD', AA' of a tetrahedron $(T)=DABC$ have a point in common, the plane $DAA'D'$ is perpendicular to each of the planes ABC, DBC, and is therefore perpendicular to the line of intersection BC of the latter two planes. Hence the line BC forms a right angle with DA, which proves the proposition.

Conversely, *if the edge* DA *is perpendicular to the edge* BC, *the altitudes* DD′, AA′ *meet in a point.*

Indeed, since DA is perpendicular to BC, a plane may be drawn through DA perpendicular to BC, and that plane will contain the perpendiculars DD', AA' dropped from the points D, A upon the planes ABC, DBC, respectively. Hence the proposition.

As an immediate corollary of the direct proposition we have this theorem.

THEOREM: *If the four altitudes of a tetrahedron have a point in common, each edge is perpendicular to the opposite edge.*

We may also readily establish the converse theorem.

CONVERSE THEOREM: *If each edge of a tetrahedron is orthogonal to the opposite*

* Numbers within brackets refer to the References at the end of the article.

edge, the four altitudes of the tetrahedron have a common point.

Indeed, by the converse proposition proved above, any altitude will have a point in common with each of the remaining three altitudes, which can take place only if the four lines considered have a common point, or if all four lie in the same plane. But the latter alternative must be excluded, since that would imply that the four vertices of the given tetrahedron lie in the plane of the four altitudes.

A tetrahedron whose altitudes meet in a point is said to be *orthocentric*. The regular tetrahedron is a conspicuous special case of an orthocentric tetrahedron.

If a perpendicular is erected to the plane of a triangle ABC *at the orthocenter* H of that triangle, and D is any point of that perpendicular, each edge of the tetrahedron $DABC$ is perpendicular to the opposite edge, and the tetrahedron is therefore orthocentric.

These are the elementary considerations that provide an answer to our geometrical question—an answer stated in simple terms.

There remains the historical question: Had the Greeks known this property? It would be utterly absurd to say that they were not capable of devising an argument such as the one just considered. Nevertheless, no mention of this proposition and no allusion to it are to be found in their writings. Why not? Why did not Archimedes himself consider it? This is the puzzle. And the puzzle only thickens if we observe that the same proposition eluded the mathematicians of the Arabic period as well as those of the Italian Renaissance.

Not until the seventeenth century, with the advent of analytic geometry, do we meet the special case of the orthocentric tetrahedron formed by the three co-ordinate planes and an arbitrary fourth plane. In this case the origin of co-ordinates is both a vertex of the tetrahedron and the orthocenter, since the edges passing through that point are also the altitudes of the tetrahedron.

The basic condition which a tetrahedron must satisfy in order that its altitudes shall be concurrent was first recognized by S. A. J. Lhuilier (1750–1840) [5].

The proof given above has been known for a century and a half [6], but up to the present it has found its way into very few books. Will the proposition, to become well known, have to wait as long as the corresponding proposition for the triangle? Practically all present-day textbooks on plane geometry include the "theorem of Archimedes." But that is a recent innovation. The famous revision of Euclid's *Elements* by A. M. Legendre (1752–1833), widely used as a textbook during the first half of the nineteenth century both in Europe and in America, ignored that proposition. Neither could the American mathematician William Chauvenet (1820–1870) find room for it in his ambitious, for the time, *Treatise on Elementary Geometry* "for colleges and schools" (Philadelphia, 1869, 1881).

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Editor's note. The interested reader may also care to read N. A. Court, "The Tetrahedron and Its Altitudes," *Scripta Mathematica*, XIV (1948), 85–96; N. A. Court, *Modern Pure Solid Geometry* (New York: The Macmillan Company, 1935), pp. 61–71; and N. A. Court, "Notes on the Orthocentric Tetrahedron," *The American Mathematical Monthly*, XL (October, 1934), 499–502. The last reference may be consulted for a bibliography on the orthocentric tetrahedron.

The earliest symbol in mathematical logic

by Howard Eves, University of Maine, Orono, Maine

Although Gottfried Wilhelm Leibniz (1646–1716) is generally regarded as the founder of symbolic logic, some mathematical symbols of a logical, rather than of an operational, nature were employed before him. Thus Pierre Hérigone published, in Paris in 1634, a six-volume work on mathematics called *Cursus mathematicus*, in which he employed an extensive set of mathematical symbols of both the logical and the operational type. But his logical symbols were merely abbreviations of certain useful phrases. Thus he used "hyp." as an abbreviation for "from the hypothesis it follows," and "constr." for "from the construction one has," etc.

Perhaps the first genuine symbol of a logical nature, and one that has lasted into present times, is the familiar three dots, $\therefore$, for "therefore." This symbol was introduced by Johann Heinrich Rahn (1622–1676), a Swiss mathematician who wrote in German, in his *Teutsche Algebra*, published in Zurich in 1659. In fact, in the *Teutsche Algebra* we find both $\therefore$ and $\because$ appearing for "therefore," the former symbol, however, predominating.

Rahn's treatise had only a minor influence in continental Europe. But in 1668 an English translation of the work, entitled *An Introduction to Algebra*, appeared in London, and this translation considerably influenced subsequent British writers. The translation was made by Thomas Brancker (1636–1676) and was edited with alterations and additions by John Pell (1611–1685). As in the original treatise, both $\therefore$ and $\because$ appear as symbols for "therefore," but now $\because$ is predominant.

It is interesting to note that in Rahn's book, and in many later British texts of the eighteenth century, the two three-dot symbols for "therefore" were used principally in passing from a proportion to the equation obtained by equating the product of the extremes to the product of the means.

Today, in Great Britain and the United States, the symbol $\therefore$ is commonly used for "therefore," and the symbol $\because$ has come to mean "because." This differentiation in the meanings of the two symbols apparently did not occur until the beginning of the nineteenth century, and the symbol for "because" has not met with as wide an acceptance as that for "therefore." Both symbols are rarely found in continental European publications except in some on symbolic logic.

The difference in influence of the original treatise by Rahn and of the English translation of the work may also be noted by the fact that though Rahn also first introduced the symbol $\div$ for division, this symbol is so used today only in Great Britain and America. The symbol $\div$ is commonly used in continental Europe to indicate subtraction.

Another historical curiosity can be traced to the English translation of Rahn's treatise. Among the additions made by Pell to the Brancker translation was a laborious treatment, due jointly to John Wallis (1616–1703) and Lord Brouncker (ca. 1620–1684), of the diophantine equation $ax^2+1=y^2$, where a is a nonsquare integer. Although Pell had no other connection with the equation than this, because of a misunderstanding on the part of the great Swiss mathematician Leonhard Euler (1707–1783), the equation has become known in number theory as *the Pell equation*.

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Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

As mathematics teachers we're foundation engineers

by Edwin Eagle, San Diego State College, San Diego, California

Engineers are sometimes faced with the problem of a building foundation that has become incapable of carrying its load. In some cases, because of poor engineering in initial stages, an adequate foundation had never been built. In other cases, for one reason or another a foundation that once was adequate has now deteriorated. In any case, adding more stories to the building or installing heavier equipment is impossible unless the foundational defects are corrected.

In actual practice, engineers have found that they need not tear down the superstructure because of this. By drilling holes and pumping liquid concrete deep into the ground beneath, they can develop an entirely adequate foundation. This process they call "grouting."

We mathematics teachers at all grade levels face situations analogous to these problems of the "foundation engineers." We find in the minds of our students some mathematics structures for which no satisfactory foundation has been previously built; others where reasonably good foundations have deteriorated, and others where heavier demands require a stronger foundation than had been anticipated. We can expect this situation to continue. At each grade level we should consider the strengthening or supplying of foundations of understanding for processes supposedly covered in previous grades an unavoidable, and even normal, part of our job.

As the present curriculum is structured, a major responsibility for checking and re-establishing foundations falls on the junior high school teacher. We must be forever "grouting." We must be "foundation engineers," ever inspecting to see what foundation exists, what condition it is in, and how adequate it is in terms of the current and future requirements. A superstructure on a superficial and inadequate foundation need not represent a total loss. Quite possibly, at considerable saving of time and effort, the structure may be retained intact and a foundation adequate for effective functional use and capable of supporting additional mathematical structure can be "grouted" under it. When, from time to time, this is successfully achieved it is a very rewarding experience for teacher and pupil. A few typical topics of mathematics that students often learn superficially, and that require considerable "grouting" may be considered as illustrations.

The place-value principle and how it operates in the algorithms for the four fundamental operations is a topic fundamental to the understanding of arithmetic processes. Yet it typically is handled very superficially or is neglected in junior and senior high school. With most students

this is a case of serious deterioration of a foundation that was not very strong in the beginning. We cannot expect students at the junior and senior high level to be as interested in counting toothpicks and putting them in bundles of ten, in putting bundles of tickets in place-value pockets, etc., as they were in the elementary grades. But we can recall these procedures, using either chalkboard diagrams or the actual devices, and covering the material in less time than would be possible with younger children. We can also have pupils use their knowledge of our money system to help them understand place value, realizing that the numbers in the units column can represent cents and those in the tens column, dimes.

On this basis the four fundamental processes with one-digit numbers and two-digit numbers can be rationalized. Adding $7+5$ can be portrayed by putting 7 pennies alongside of 5 pennies. Ten of the pennies may be exchanged for 1 dime, leaving 2 pennies. The resulting sum, 1 dime and 2 pennies, may be indicated by the number 12. Subtracting 23 from 43 can be portrayed as follows:

Given	43 cents,	4 dimes, and	3 pennies,
subtract	25 cents,	2 dimes, and	5 pennies.

Since we cannot take 5 pennies from 3 pennies, we exchange one of the 4 dimes for 10 pennies, and we have then

	3 dimes	and	13 pennies.
We are to subtract	2 dimes	and	5 pennies,
which leaves	1 dime	and	18 pennies.

By actually "taking away," counting away 5 pennies one at a time and then 2 dimes, we would find a remainder of 1 dime and 8 pennies, which we may write as 18.

For illustrating operations with numbers of more than two digits, "dollar units" are preferable to "cent units." Dollar units mean the tens column will contain ten dollar bills, the hundreds column hundred dollar bills, etc. We can also consider the decimal fractions, with dimes in the tenths column and cents in the hundredths column.

The metric system of linear measurement is another useful device for illustrating in a visual manner our base-ten number system. This is one of the important reasons for teaching the metric system, and this desired outcome should be kept clearly in mind when the topic is studied.

Having established, or re-established, an understanding of the place-value principle and its operation in addition and subtraction, multiplication can be rationalized as repeated additions. Division is understood as repeated subtractions. To illustrate the latter, let us consider $9775 \div 23$. This means that we are to find how many 23's are contained in 9775. One way to find this would be to repeatedly subtract 23 from 9775, and see how many times we could subtract it before arriving at zero, or at some number less than 23. This operation could be visualized as repeatedly cutting off 23-inch lengths from a ribbon 9775 inches long. Since this would be a long, tedious process, we subtract 23 in multiples of 100 and 10. We can successively subtract four hundred 23's, twenty 23's, and five 23's as follows:

Starting with	9775
Subtract 400 23's =	9200
Leaving	575
Subtract 20 23's =	460
Leaving	115
Subtract 5 23's =	115
Leaving	0

Quite evidently there are 425 23's contained in 9775.

From this it is only a matter of rearranging to get the following:

$$\begin{array}{r} \left.\begin{array}{r} 5 \\ 20 \\ 400 \end{array}\right\} 425 \\ \hline 23\overline{)9775} \\ 9200 \\ \hline 575 \\ 460 \\ \hline 115 \\ 115 \\ \hline 0 \end{array}$$

This in turn can be condensed to the algorithm:

$$\begin{array}{r}
425 \\
23\overline{)9775} \\
\underline{92} \\
57 \\
\underline{46} \\
115 \\
\underline{115} \\
0
\end{array}$$

When students have done a few division problems in which they use all three forms for each problem, they soon realize that the standard long-division algorithm is merely a concise arrangement for doing repeated subtractions to determine how many times the divisor is contained in the dividend.

Many students see little reason for trying to rationalize a sequence of steps that have become automatic. One device which will arouse considerable interest, and also stimulate students to think through the place-value principle and its operation in calculating, is to have them develop and actually use a number system having a base other than 10. Among a semiprimitive New Zealand tribe the word for 4 is dog; apparently the "fourness" of the dog's legs is an impressive feature. Adapting our Hindu-Arabic symbols and system to the base-four idea of the New Zealanders and modifying the names to avoid confusion, we might develop a "Hin-dog-abic" number system in which the first 20 numbers would be as appears in the second column on this page.

The system can, of course, be extended indefinitely. The number 1000 in base four, doggy houndred, might be called one kennel, just as ten hundred may be called one thousand in our base-ten numerals. "Dogcil" fractions may also be included. One dogth, 0.1, would equal the one fourth of base-ten numerals, and 0.01 would equal the one sixteenth of base-ten numerals. Checking would be done by "casting out" threes instead of by "casting out" nines.

Hindu-Arabic Numeral	*Hin-dog-abic Numeral*	*Hin-dog-abic Number Word*
0	0	zero
1	1	one
2	2	two
3	3	three
4	10	doggy
5	11	doggy-one
6	12	doggy-two
7	13	doggy-three
8	20	twoggy
9	21	twoggy-one
10	22	twoggy-two
11	23	twoggy-three
12	30	throggy
13	31	throggy-one
14	32	throggy-two
15	33	throggy-three
16	100	one houndred
17	101	one houndred one
18	102	one houndred two
19	103	one houndred three
20	110	one houndred doggy

Students can undertake extensive projects developing and using number systems having bases other than 10. They begin by writing in sequence enough numbers to arrive at three-digit numbers, give names to the numbers, and learn to count. They then work out their own tables of the addition and multiplication facts, perform calculations and check their answers, and compare the processes with the corresponding processes in our base-ten system. They even design systems of linear, area, and volume measurement, and systems of measure for time, temperature, weight, money, etc. Designing, making, and using an abacus for the new number system are valuable in portraying the facts more clearly. Such projects help students realize that number systems are man-made, and based on simple, reasonable principles rather than on arbitrary rules to be memorized and blindly followed. The students' foundations for further work in arithmetic are greatly strengthened by the new point of view and by the experiences of problem solving, analyzing, and comparing that such projects entail.[1]

[1] For a more detailed discussion of numbers to other bases and for an example of a base-five number system see Marks, J., Purdy, C., and Kinney, L., *Teaching Arithmetic for Understanding* (New York: McGraw-Hill, 1958).

Another process that is poorly performed and even more poorly understood is division by a common fraction. Even for students who do get the correct answer to such problems, the answer is ordinarily the meaningless result of a memorized rule, "Invert the divisor and multiply." This rule can become meaningful if students are encouraged, or required, to improvise their own solutions and to use their own improvised procedures until, with whatever guidance is needed, they discover the rule for themselves. To illustrate consider the following teacher-pupil conversation (much condensed) which could well occur in a classroom at any level from grade 6 through grade 14. Note that, starting with concrete problems, the pupil can improvise procedures to get the answer without necessarily realizing that he is doing division problems. (See Figure 1.)

TEACHER: How many half-foot lengths of ribbon can be cut from this 4-foot length of ribbon?

PUPIL: Eight pieces.

T.: How did you get the answer?

P.: There would be 2 half-foot lengths in each foot, so there would be 8 half-foot lengths in 4 feet.

T.: What process did you use to get this answer?

P.: I multiplied 4×2.

Other concrete examples with fractions having unit numerators would then follow. Then, returning to this problem:

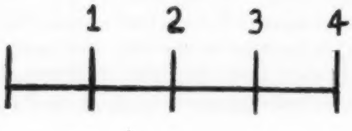
T.: Is this a division problem?

P.: Well, I got the answer by multiplying.

T.: Is finding how many times a small piece is contained in a large piece an example of division?

P.: Yes, I guess it is.

Figure 1



T. If we restate the problem using inches, it becomes, "How many 6-inch pieces of ribbon can be cut from a ribbon 48 inches long?" Solve it, using inches.

P.: $48 \div 6$ is 8. There are 8 pieces.

T.: Correct, and the same answer you got before:

48 inches divided by 6 inches equals 8;
4 feet divided by 1 half-foot equals 8.

This is the same division problem expressed in two different ways. But you got your answer by multiplying 4×2. Where did you get the 2?

P.: There are 2 half-foot lengths in one foot.

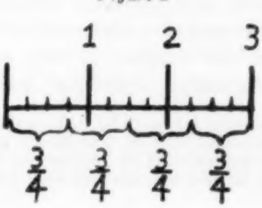
T.: Right. Now note that if we invert $\frac{1}{2}$ we get 2/1 or 2. (This last stage should not be approached too quickly.)

It follows from the fact that we agreed to use the symbol $\frac{1}{2}$ to represent a piece of such size that 2 such pieces would equal 1 unit. So when you invert the divisor you find how many times it is contained in 1 unit. Then to find how many times it is contained in 4 units you multiply by 4. (These ideas are developed by exploration and demonstration.)

For some time, even when reviewing such processes, students should use their own improvised procedures based on meanings. If they move too quickly to the manipulation of symbols on the basis of memorized rules the meanings become difficult to see and are seldom really learned. After considerable practice with problems having divisors with unit numerators, problems with numbers other than 1 can be considered. Diagrams and explanations similar to the following may be used.

$3 \div \frac{3}{4}$ may be pictured as in Figure 2.

Figure 2



The picture also shows that the divisor is contained in 1 one and one-third times (one divisor and one third of another divisor) or 4/3 times or 4 times. The problem may also be pictured as finding how many $\frac{3}{4}$-pie-portions can be served from 3 pies. To verify the answer by a different method the problem may be considered as 3 dollars divided by $\frac{3}{4}$ of a dollar and handled as \$3.00÷.75.

A number of problems should be treated in this fashion, in each case analyzing by means of real objects, rulers, or diagrams; determining the answer by counting the pieces; and then noting that inverting the divisor and multiplying gives the correct result. Also it is necessary to observe repeatedly that the inverted divisor tells us how many times the divisor is contained in 1. On the basis of some such approach as this, the entire process is made to seem sensible to the student and a better foundation for further progress is built.

Among the concepts that are vague and fuzzy among secondary pupils is that of area. Most of them can glibly recite and use $A=bh$, and recall that the base times the height gives the area of a rectangle. Having acquired the notion that "feet ×feet = square feet," they cannot clearly visualize area as "the number of square units." Finding area by actually counting squares will rebuild, or build for the first time, this basic concept of area. When the topic is first considered, and in the early part of review lessons, the students should be instructed to find "the number of squares," or "the number of square units."[3] Using squared paper, the students can find the area of various geometric figures by counting squares, estimating fractional parts of squares where necessary. After some exercises with rectilineal figures the area of a circle can be found by counting the squares in one quadrant, again estimating fractional parts, and

[3] In *Teaching Arithmetic for Understanding, op. cit.*, it is recommended that the introduction of the word "area" be delayed until the concept has been established and the need for some such word sensed by the students.

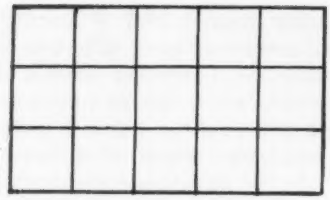


Figure 3

multiplying by 4. Using maps, the areas of lakes, cities, counties, etc., can be determined. For this purpose U. S. Forest Service maps, which are ruled off in square miles, are useful.

When the concept of area as "the number of squares" has been sharpened, the rule and the formula for finding the area of a rectangle becomes understandable. To illustrate with the rectangle pictured in Figure 3, the student may progress through the following stages.

1. By counting all of the squares arrive at a total of 15 square units.
2. Noting 5 squares in the top row, 5 in the middle row, and 5 in the bottom row, add $5+5+5$ to get 15 square units.
3. Noting 3 squares in each column and 5 columns, add $3+3+3+3+3$ to get a total of 15 square units.
4. Recalling that multiplication is a short way of doing repeated additions, see that item 2 becomes $3\times5=15$, and that item 3 becomes $5\times3=15$. (This is a nice visual illustration of the commutative law of multiplication, which should receive considerable comment from time to time.)
5. Generalize, seeing clearly that in any rectangle the length always indicates the number of unit squares in one row and the height the number of rows, so that the product indicates the total number of squares. (Similarly, that the height shows the number of unit squares in one column and the base, the number of columns.)
6. With understanding based on the above steps be able to give a clear statement of the rule for finding the area of a rectangle.
7. Understand the formula $A=bh$ as a concise way of expressing the idea that has already been understood and expressed in other ways.

The concept of area as "the number of square units" is so basic that it should be kept alive in the student's mind. He will need it in order to develop and apply the other area formulas with understanding, and in order to move on to more difficult

concepts such as, "Areas of similar figures vary directly as the squares of corresponding linear dimensions." Properly directed review from time to time is necessary to preserve and strengthen the concept, even where it has been quite effectively developed when first introduced.

Howard Fehr[3] comments that the processes of mathematics are easy to teach and easy to test, but that an understanding of the concepts of mathematics is difficult to teach and even more difficult to test. Probably this accounts in large part for the fact that too often we mathematics teachers tend to do the easier thing and to teach little more than the processes. On the basis of satisfactory pupil performance on various types of tests, we may then delude ourselves into thinking that we have done an effective and adequate job of teaching. But we must do a more thorough job. We must not be guided too much by short-term goals and expediencies of the moment. Fehr goes on to point out that though teaching an understanding of the concepts is more difficult and at the time a slow process, in the long run it actually saves time, due to better retention and greater facility at later steps along the way.

So at each stage as we help each student build his mathematics structure we must be mindful of the foundation, and not of the walls and ceiling only. To change the analogy, we must see that the roots spread out and grow deeper as the branches spread and as the trunk grows taller. That is the natural way for the tree to grow.

[3] Howard Fehr, "Secondary Mathematics, Functional Appeal."

Have you read?

BROWN, JOHN A. and MAYOR, JOHN R. "Preparation of the Junior High School Mathematics Teacher," *The Journal of Teacher Education*, June 1958, pp. 142–148.

Mathematics is changing and the people to whom it is taught are also changing. The junior high school is a growing concern. What should be the preparation of the mathematics teachers who are soon to man the junior high school classrooms? The authors here present a rather detailed outline of what they think this preparation should be. It includes adolescent psychology, the psychology of learning, student teaching, and methods of teaching in the area to be taught. Every teacher should have a general education in humanities, social science. biological science, astronomy, and other fields.

But in what areas should the major in mathematics be prepared? These areas should include analysis, foundations of mathematics, modern algebra, statistics, advanced geometry, applications to mechanics, and the theory of games. In case you know someone who feels that the people preparing teachers do not believe in subject matter, ask him to read this article. Most teachers trained in the average college ten years ago, with no work since, could not meet the authors' specifications. Read the article and see if you do!—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

HIRSCHI, L. EDWIN. "An Experiment in the Teaching of Elementary Algebra," *School Review*, Summer 1958, pp. 185–194.

As has often been said, modern mathematics is more a change in approach than a change of content. Mr. Hirschi has tried just such a change. His experiment involved 180 students and three teachers. The design was one control group with traditional presentation and an experimental group with a different approach. The control group was ultratraditional both as to instruction and class atmosphere. The experimental group was informal; each student was in competition with himself; the discovery of knowledge was encouraged; teaching procedures grew out of students' needs; the textbook was brief, with few rules, and it emphasized broad concepts. The article explains the experiment more fully, but what were the results?

Both groups did equally well on a standard test. A second test validated by a panel was also given, and the experimental group made higher scores than the control group.

What do these results mean? Perhaps we should also look at the content and see if its order and presentation can be justified. Read the article and decide this question for yourself. —PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Modern mathematics in a summer high school course

by James E. Stochl, University of Minnesota High School Minneapolis, Minnesota

One way of introducing high school students to modern mathematics is to offer a summer school course. It is also a means of offering enrichment for the gifted student. For example, during the past two years the University High School at the University of Minnesota has offered a special course in mathematics as a part of its regular summer session program. This course, called "A Survey of New Mathematics," was designed for high school pupils with high interest and aptitude in mathematics. Classes met two hours a day for five weeks during June and July with an instructor from the regular staff of the high school.

When the course was set up, it was decided that pupils should have completed plane geometry and have at least a B average in mathematics. During the summers of 1957 and 1958, 33 students registered for this course. Eighteen of them had completed the tenth grade and 15 had completed the eleventh grade. Eighteen of them had straight A averages. These 33 students represented 17 different high schools: 8 large metropolitan schools, 6 suburban schools, and 3 small-town schools.

To say that these young people were interested in mathematics would be an understatement. This is shown by the fact that even though no high school or college credit was given for satisfactorily completing this course, the students came to school during the warmest part of the summer because they liked mathematics. All of them commuted some distance from their homes to the University campus in Minneapolis each day. In fact, one student in 1957 and another in 1958 drove daily from a town more than 50 miles from Minneapolis. The first day of class, they were asked to fill in answers to a questionnaire. The following is a typical answer to the question, "Why do you want to take this course?" "I like mathematics and, besides, I feel I need as much mathematics as I can get for the occupation I plan on entering."

The course itself was a survey of many topics, since it was not our purpose to teach any one topic completely. It included ideas from topics such as Binary Number System, Symbolic Logic and Truth Tables, Fourth Dimension, Theory of Games, Finite Geometry, Sets and Set Notation, Boolean Algebra and Switching Circuits, and Modern Computing Machines. These topics were presented in lecture form by the instructor and were followed by class discussion and individual assignments. A great deal of reference and supplementary material was given to

the students in mimeographed form. A very complete collection of reference books was also available in the high school mathematics library. The reference books used as textbooks were: *Principles of Mathematics* by Allendoerfer and Oakley, *Introduction to Finite Mathematics* by Kemeny, Snell, and Thompson, and *Fundamentals of Mathematics* by Richardson.

Each student was required to work on an individual project or report on a topic of interest to him. These reports were then presented to the whole class during the last week. Individual initiative and project uniqueness were encouraged. Not only were some excellent projects developed, but some of the students continued to work on their projects after the course had been completed.

On the last day another questionnaire was distributed to determine the pupils' reaction to the course. Answers to the questions "What topic did you like most?" and "What topic interested you least?" were too varied to summarize. Because the course was being offered on a trial basis, these two questions were asked: "Would you recommend teaching this course again next year?" and "If it is offered, would you suggest it to your friends?" The answer to each was a unanimous *yes*. The students were then encouraged to comment freely about the course. Their reactions were most favorable. A few sample comments are quoted here:

> "I really enjoyed it, and thought it was always interesting. I'm sorry to see it end."
>
> "I enjoyed this course in general because the things which are taught in it are not easily obtainable by any other means."
>
> "Add some more time, maybe eight weeks instead of five."
>
> "I thought the class was very worthwile, and if I had a chance to take a similar one, I would."

It would seem obvious that "A Survey of New Mathematics" course is worth offering in any high school summer program.

Graphing pictures

by Margaret L. Carver, McAllen High School, McAllen, Texas

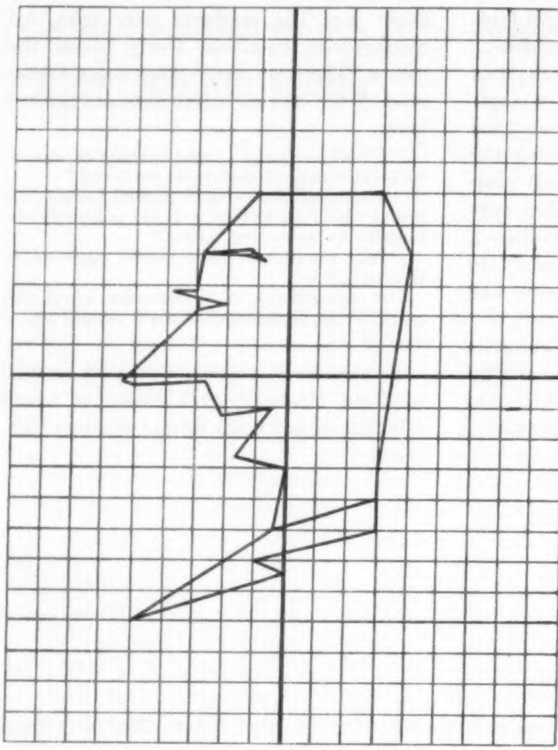
Graphing can be made more palatable to junior high school students, even beginners, by assigning exercises which produce pictures.

My students have been enthusiastic about graphing pictures as the location and connection of points located by coordinates. Students can produce a 30-point picture in 10 to 15 minutes and quickly see their errors as defects in the picture. Although this work is assigned for only one day's classwork, many students ask for additional lists of picture-graph points to work on at home, and some amuse themselves outside of class by devising their own picture graphs.

Picture possibilities are limited only by the inventiveness of the teacher and students. Cartoon-type pictures are usually more appealing and easier to adapt to graphing. Almost any simple line drawing can be graphed. The lively kangaroo was "borrowed" from an advertisement of Qantas (cq) Airlines of Australia. The "prof" and the cat were inventions. After students have studied comic sections,

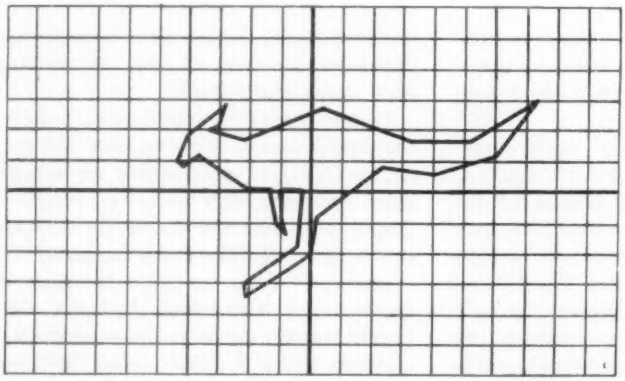
many additional possibilities are readily available.

The author's method consists of distributing duplicated lists of the points and the announcement that lines connecting the points serially will produce pictures. Curiosity, followed by amusement, a sense of accomplishment, and a measure of self-criticism as to accuracy and neatness, does the rest.



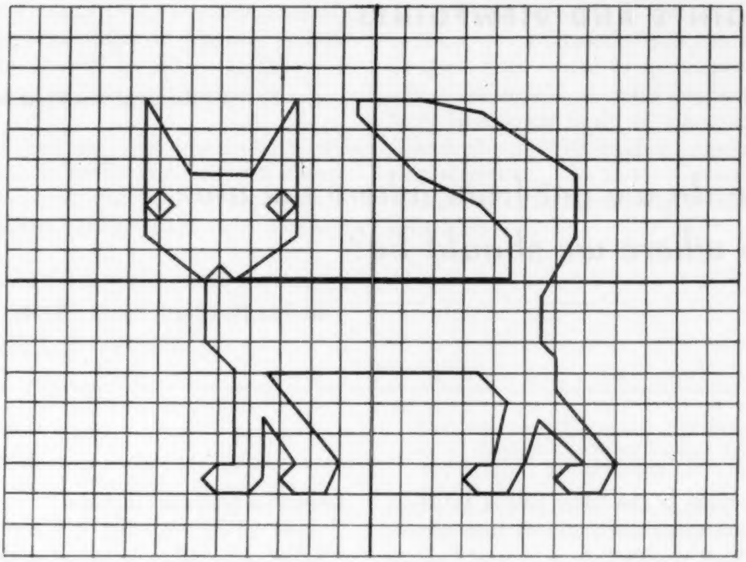
The prof

x	y	x	y
15	−20	−14	20
− 2	−25	−15	14
−25	−40	−19	14
0	−32	−10	12
− 5	−30	−15	11
15	−25	−15	13
15	−15	−27	− 1
20	20	−25	− 2
15	30	−13	− 1
− 5	30	−11	− 7
−14	20	− 2	− 5
− 6	21	− 8	−13
− 4	19	0	−15
− 6	20	− 2	−25



The kangaroo

x	y	x	y
−14	14	−11	−15
−15	11	− 2	− 9
−17	10	− 1	0
−11	8	− 5	0
2	13	− 5	− 4
17	8	− 4	− 7
26	8	− 6	− 5
37	15	− 7	0
30	6	−10	0
20	3	−18	6
12	4	−21	4
1	− 4	−22	5
0	−10	−20	9
−11	−17	−14	14



The cat

x	y	x	y	x	y	x	y
−9	0	7	−6	11	−4	−14	6
−11	0	9	−8	11	1	−13	5
−11	−4	8	−12	13	3	−14	4
−9	−6	7	−12	13	7	−15	5
−9	−12	6	−13	7	11	−15	12
−10	−12	7	−14	3	12	−12	7
−11	−13	9	−14	−1	12	−8	7
−10	−14	10	−12	−1	11	−5	12
−8	−14	11	−9	3	7	−5	5
−7	−13	14	−12	7	5	−6	4
−7	−9	13	−12	9	3	−7	5
−5	−12	12	−13	9	0	−6	6
−6	−13	13	−14	−9	0	−5	5
−5	−14	15	−14	−10	1	−5	3
−3	−14	16	−12	−11	0	−9	0
−2	−12	12	−7	−15	3		
−7	−6	12	−5	−15	5		

Purdue—General Electric Fellowship Program

Purdue University announces the seventh annual all-expense General Electric Summer Fellowship Program in mathematics for junior and senior high school teachers from June 22 through August 1, 1959. Courses in the program carry graduate credit. Eight semester hours may be earned. Any secondary school teacher of mathematics who teaches in Illinois, Indiana, Iowa, Kentucky, Michigan, Minnesota, Missouri, Ohio, Tennessee, West Virginia, or Wisconsin, has a bachelor's degree, has had differential and integral calculus, and has not previously held a General Electric Summer Fellowship in Mathematics or Science is eligible.

This program is sponsored by the General Electric Educational and Charitable Fund in cooperation with the Department of Mathematics and Statistics. Further information and application forms may be obtained by writing to:
General Electric Summer Fellowship Program,
Department of Mathematics and Statistics,
Purdue University, Lafayette, Indiana

A column of unofficial comment

How do we get from where we are to where we should be?

by Ida Bernhard Puett, Atlanta, Georgia

The needs of the latter half of this century have made it mandatory that schools and school people take a critical look at present mathematics offerings. Many states are experimenting in various areas of the mathematics curriculum. Studies are being conducted on a state-wide level as well as in local school systems. In pursuing these studies, attention is being focused upon such projects as those undertaken by the Commission on Mathematics of the College Entrance Examination Board, the University of Illinois Committee on School Mathematics, the University of Maryland Study for the Junior High School, and the School Mathematics Study Group.

In attempting to move into programs incorporating the newer mathematics, schools must answer the question, "How do we get from where we are to where we should be?" To accomplish this, improvements must be made in preservice, in-service, and administrative practices in our schools.

Preservice education practices needing consideration include:

- reorganization of present teacher education programs in mathematics as to content of subject matter, methods courses, laboratory experiences, and summer programs for preservice teachers.

In-service practices include:

- more careful planning of in-service education programs—not only those within a given school system but also those under the leadership of colleges for which college credit is given. (They should be based upon the needs of the teachers in relation to their teaching assignments.)

Administrative practices include:

- more encouragement by administrators in furthering growth of teachers in their knowledge of modern mathematics:
  - learning about and encouraging teachers to obtain summer and academic year scholarships; obtaining financial assistance for teachers;
- a more intensive look by local schools at each of the offerings in mathematics as to content, particularly with regard to selection of textbooks;
- additional money for materials in mathematics such as library, films, and appropriate visual aids, including the materials for making them (this is assuming that teachers obligate themselves to make careful selections and evaluations of such materials);
- additional experimentation through cooperative efforts of local schools, colleges, and universities;

making use of every resource available to teachers—those in business and industry as well as others in the community, the state, and the nation.

Many schools, and particularly the small schools, are having a difficult time in keeping abreast with new developments in mathematics. Local, state, and national committees concerned with curriculum revision in mathematics will not be able to move from where we are at present to where we should be until improvements have been made in all of the areas mentioned above. Will teachers, supervisors, administrators, and those responsible for teacher education meet the challenge we now face?

What's new?

BOOKLETS

How to Study, How to Solve (2d edition), Addison-Wesley Publishing Company, Inc., Reading, Massachusetts. 43-page booklet written by H. M. Dadourian, 50¢ each.

Junior-Year Science and Mathematics Students, Fall 1957 (Catalog No. FS 5.4:520), Superintendent of Documents, Government Printing Office, Washington 25, D. C. 56-page booklet giving data of the first nation-wide survey of junior-year men and women majoring in various fields of science or mathematics in colleges and universities, 45¢.

Mathematics and Science Before College, Dwight L. Arnold, Kent State University, Kent, Ohio. 12-page booklet, free.

Mathematics Clubs in High Schools, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C. 32-page booklet edited by Walter H. Carnahan, 75¢.

Mathematics for Today, Seattle Public Schools, 815 Fourth Avenue North, Seattle 9, Washington. 30-page illustrated booklet dealing with the mathematics program in grades 1–12 in the Seattle public schools, 65¢ each.

Modernizing the Mathematics Curriculum, Sets, Relations, and Functions, The Introduction to Algebra from the Point of View of Mathematical Structure, Executive Director, Commission on Mathematics, College Entrance Examination Board, 425 West 117th Street, New York 27, New York. Three separate booklets, the first of which is addressed to superintendents of schools and members of school boards; free.

Money Management: Your Health and Recreation Dollar, Money Management Institute, Household Finance Corporation, Prudential Plaza, Chicago 1, Illinois. 36-page illustrated booklet, 10¢ each.

Opportunities Unlimited for Physicist, Engineer, Mathematician, Electronic Scientist, Chemist, and Metallurgist (Announcement No. 5-35-1) U. S. Civil Service Commission, Bureau of Departmental Operations, Washington 25, D. C. 32-page illustrated booklet describing employment opportunities at Redstone Arsenal, free.

Preparation of Teachers for Secondary Schools, National Council of Independent Schools, 84 State Street, Boston 9, Massachusetts. 52-page booklet; single copies free; additional copies 25¢ each.

Professional Opportunities in Mathematics (3d edition), The Mathematical Association of America, University of Buffalo, Buffalo 14, New York. 24-page booklet, 25¢.

Recreational Mathematics (2d edition), National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C. 143-page booklet written and compiled by William L. Schaaf giving bibliography on this topic, $1.20 each.

Start Planning Now for Your Career, Department 2-119, General Electric Company, Schenectady, New York. 7-page illustrated booklet, free.

Teaching Taxes Kit, Internal Revenue Service, Washington 25, D. C. 1958–59 editions of kits available for teaching about income tax returns, free.

Time: Past, Present, Future, Paragon Electric Company, Two Rivers, Wisconsin. 32-page booklet (comic-book style) dealing with instruments for measuring time; reasonable quantities free to schools.

Updating Mathematics, Arthur C. Croft Publications, 100 Garfield Avenue, New London, Connecticut. Monthly newsletter made up of separate sections for administrators, elementary mathematics teachers, junior high mathematics teachers, and high school mathematics teachers; 9 complete letters, $30; additional subscriptions for letters to teachers, $5 per year per section.

World of Numbers, International Business Machines Corp., 590 Madison Avenue, New York 22, New York. 18-page illustrated booklet dealing with numbers and digital computers, free.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Analytic Geometry Problems, C. O. Oakley (New York: Barnes and Noble, Inc., 1958). Paper, xvii+253 pp., $1.95.

This book was prepared to provide a group of problems that typify those found in the study of Analytic Geometry. The theory in the book has been intentionally reduced to explanation of fundamental principles only. It could possibly serve as a text for the better student. For the majority it would be advisable, as the author suggests, to use it in conjunction with another textbook. The author has listed 25 standard Analytic Geometry textbooks with cross references for various topics found in this book of problems.

There are approximately 600 problems in the book, with descriptive solutions given for 341 of them. Answers are given for 345 of the problems. There is also an appendix that contains brief tables of squares, cubes, square roots, cube roots, common logarithms, natural logarithms, trigonometric functions, and values of e^x and e^{-x}.

Direction cosines and direction numbers are defined for a line in two dimensions early in the book (p. 11–12). It seems better to introduce this concept here rather than to wait until the section on solid analytic geometry is reached. The ellipse and hyperbola are defined in terms of the eccentricity. The properties of the focal radii to a point on the curve sometimes used to define the ellipse or hyperbola are taken up as problems. Transformations of co-ordinates are taken up after work on the conic sections has been completed. There are separate chapters on "Poles and Polars" and "Diameters," preceding the section on polar co-ordinates. The chapter on higher plane curves is especially good as reference material to be used in addition to that given in any standard text. No mention is made of vectors or their use anywhere in the book, although there is a definition of directed line segments in the chapter on polar co-ordinates. Though brief, the theory in the book appears to be mathematically sound, and the problems are good.

The author suggests that this book be used by itself to enable a student to get an understanding of analytic geometry, or be used along with a standard textbook. It seems to us, however, that a book of this type would be more useful as a reference for students who have already completed a course in analytic geometry. —*Alan H. Paine, Potomac State College of West Virginia University, Keyser, West Virginia.*

Basic Geometry (reprint), George David Birkhoff and Ralph Beatley (New York: Chelsea Publishing Company, 1958). Cloth, 294 pp., $3.95.

By departing radically from the Euclidean axiom-scheme, and by making free use of the properties of real numbers, the authors have been able to give a set of axioms that are intuitively acceptable and yet lead very quickly to the basic theorems of plane Euclidean geometry. Once this point has been reached, the development of such topics as the theory of circles, regular polygons, loci, ruler and compass constructions, cartesian co-ordinates, etc., follows with relative ease.

In addition to having the pedagogic value of getting rapidly to the more interesting results, the treatment is on a much sounder logical basis than Euclid's. The need for dependence on the intuitive notion of superposition is avoided, as are most of the arguments depending on order relations of points and lines in a plane. There are still a few places where intuitive arguments are used as part of a proof, but these are usually pointed out in footnotes. On the other hand, there is an unfortunate lack of precision in some of the definitions, very noticeable in the treatment of angle measure.

As a textbook, *Basic Geometry* suffers from too much brevity. The various topics need much more motivation and preliminary discussion than is provided. Once the basic theorems have been proved, most of the further theorems are given as problems (with frequent hints); this is good, but one wants more than just a long list of such problems.

There is a tremendous lot of good geometry in this little book. In the hands of an expert teacher, who can supply the additional material needed, it could provide an exciting introduction to abstract reasoning and the concept of an axiomatic mathematical system. Used in an unimaginative fashion with an average class, it might be an incomprehensible bore.—*R. J. Walker, Cornell University, Ithaca, New York.*

Calculus, Walter Leighton (Boston: Allyn and Bacon, Inc., 1958). Cloth, x+373 pp., $6.95.

Probably the most outstanding feature of this text is the care with which most new concepts are introduced, particularly those of the derivative and the integral. In the first chapter the student is expected to work exercises involving tangent and normal lines, maxima and minima, and rates and velocities before formulas

for differentiation introduced in the second chapter are at his disposal. Consequently, the algebraic manipulations involved in some of these otherwise typical problems may warrant the omission of the more complicated exercises in the first chapter, especially since exercises on these same topics are included in subsequent chapters after the usual differentiation formulas have been introduced.

Also, the number of exercises is large enough and the levels of difficulty varied enough that selective assignments may be made. Answers to approximately half the exercises are printed with the exercises. This is definitely not a "cook book" type of text, and there are more exercises involving proofs than are usually found in texts at this level.

The integral is introduced early but is not "applied" in as many types of situations as is customary. Instead, there are exercises on evaluating integrals by direct application of the definition and by applying theorems. Also, there are proofs of theorems to be worked out. Volumes of revolution by the "disk" and "washer" methods are not treated. Moments of solids are omitted, but moments of arcs and laminas are considered. There are sections on work and fluid pressure, and the usual techniques of integration are considered. Also, the integral representing the area of an arbitrary surface $z=f(x, y)$ above a suitable region R_{xy} is treated fairly rigorously.

The reader of this review is perhaps getting the impression that this text is meant primarily for the well-prepared, able student. We do not feel this to be the case. There are many illustrative examples. There are extensive reviews of the trigonometric, exponential, and logarithmic functions preceding the application of calculus to these functions. There is also a short chapter on solid analytic geometry confined to the line and plane preceding the chapter on partial differentiation. The text is well written, and, although theory is emphasized, the discussion is not too abstract. However, we do have a criticism or two to offer.

First, there is a scarcity of good figures, especially in three dimensions. These are particularly valuable to the student lacking imaginative powers. The illusions attempted in Figures 94 and 108 leave something to be desired. What appear to be straight, parallel lines are used here to indicate surfaces which are not planes.

Also, we question whether presenting the excellent proof of the equivalence of the double and iterated integrals after double integrals have been evaluated by this principle is the most desirable order of presentation.—*Arnold Wendt, Western Illinois University, Macomb, Illinois.*

Introduction to the Theory of Sets, Joseph Breuer (translated by Howard F. Fehr) (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958). Cloth, viii+108 pp., $4.25.

This slim book presents a survey of Cantor's intuitive set theory. The exposition is in the spirit of Hausdorff's *Mengenlehre*, but is, of course, not so comprehensive.

The book is carefully written, and the translation is very well done. Many figures accompany the discussion, and there are plenty of illustrative examples. Answers to the exercises (about eighty in all) are given at the end of the book.

The reader of this nice little book should gain some idea of how mathematicians were struggling with set theory thirty or forty years ago. It is significant that the theorem of Stone (1937) is not even mentioned. In fact, the operation of symmetric difference of two sets is not introduced in the text. The reader will also be exposed to much that has turned out to be of little value for modern mathematics. The "arithmetic" of cardinals and ordinals is not of primary concern for much of modern mathematics. Zermelo's theorem is used nowadays most often in the form of Hausdorff's (*alias* Zorn's) maximal principle. The notions of relation and function are now near the beginning of set theory, not at the end. The "paradoxes" of intuitive set theory are now of concern mainly to mathematical logicians, not to practicing mathematicians.

This reviewer agrees with the translator that "There is now a need for a treatment of set theory in English, from a less than abstract approach, sufficiently elementary to serve as an introduction to the subject for college and high school instructors, college students, and interested laymen." But this reviewer cannot agree that "This book meets that need." What is really needed in this field (it seems to us) is not a stirring of bones long since dry, but an elementary and leisurely presentation of the material nicely summarized by J. L. Kelley in Chapter 0 of his book, *General Topology*. A good approximation to this (nonexistent) treatment is given in *Universal Mathematics—Part II* by W. L. Duren, *et al.* This last work should be consulted by the high school teacher who is anxious to belong to the set of teachers who are conversant with contemporary intuitive set theory.—*M. F. Smiley, State University of Iowa, Iowa City, Iowa.*

The scholar is that man who must take up into himself all the abilities of the time, all contributions of the past, all the hopes of the future.—*Emerson.*

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Problems for the good student

by Floyd D. Strow, Ottawa Hills High School, Toledo, Ohio

One of the problems facing any teacher of mathematics is the problem of keeping the brightest pupils from becoming utterly bored with the usual mathematics problems and the repetition needed for many pupils. Many parts of mathematics, especially the fundamental operations of arithmetic and elementary algebra, must be learned quite thoroughly if a student is to have the desired success in later work in mathematics and the physical sciences.

If only all pupils would learn at the same rate or require the same amount of practice to master the quadratic formula, for example, the teacher's job would be much easier. However, we find that some pupils can do quadratics after doing ten samples, while other pupils will need many more samples to achieve a similar mastery. Often a teacher will assign additional homework for the fast learners to keep them busy, spending more class time with the slow learners. This practice tends to penalize the bright student, and he soon learns to slow down to the speed of the class.

I have found that a better plan is to give the top students special problems that require extra effort and power. In my classes I often give these students what I call "100-point problems"; that is, for the first correct answer (sometimes for the first three or four correct answers) I will give the student a test grade of 100 per cent in my grade book. Sometimes these problems are taken home and solutions brought in the next day, with the understanding, of course, that a student is not to turn in work that is not his. The type of problems that I give will vary with the class and the type of work it is doing at that particular time.

In plane geometry I have used such problems as these:

1. Construct an equilateral triangle, given the altitude.
2. Construct a right isosceles triangle, given the hypotenuse.
3. Construct the bisector of an angle when the vertex is inaccessible.
4. Construct a line that will pass through a given point and also pass through the inaccessible vertex of a given angle.
5. Find the mid-point of a line segment, using only the compass.

For the more advanced students near the end of the course there are many more difficult constructions such as:

6. Construct a right triangle, given the radii of the incircle and circumcircle.
7. Construct a trapezoid, given the four sides.
8. Construct a triangle, given the three medians.
9. Construct a triangle, given the three altitudes.

In my algebra classes I often give as 100-point problems more difficult examples of the same type of work the class is doing. For example, when the class is working on simplification of expressions containing parentheses I often use problems of this type.

10. $3a-[2a-b+3(2a-b-\{3a+b-\overline{a-2c}+2b\})]=?$

When the class is working on fractions, problems like this one will challenge the good students:

11. $$\frac{\frac{x-5}{1}-\frac{3x-15}{x+3}}{x-1}\cdot\frac{\frac{x+4}{1}-\frac{x^2-13}{x-1}}{\frac{1}{5}-\frac{1}{x}}=?$$

In my trigonometry and solid geometry classes it is not difficult to find still other problems that will tax the ability of the very able student. By giving extra credit for these solutions the student is encouraged to do his very best. Here is one problem I have used:

12. A ball 12″ in diameter is placed in the corner of a room so that it is tangent to the floor and to two walls. A smaller ball is placed behind the large ball so that it is tangent to the large ball, the floor, and the two walls. What is the diameter of the small ball?

Quite often the really tough problems find their way home with the student, and Dad has a chance to show how good a mathematician he is. However, the problems that are most often taken home are of a different type—usually problems in logic such as the following:

13. Three men who share a room at a hotel are charged \$10 each or \$30. The proprietor, after some reflection, decides that he has overcharged them because they are sharing a room, so he gives the bellboy \$5 to return to them. The bellboy, not being able to divide \$5 into three equal parts, pockets \$2 for himself and returns only \$1 to each man. That made the room cost each man \$9, or \$27 for all three. The \$2 the bellboy kept makes \$29, yet they paid \$30. Where is the other dollar?

14. There is given a platform balance (with no weights) and 8 balls that are identical in color, size, etc., except that one ball is lighter in weight than the other 7. It is required to pick out the light ball in two balancings of the platform scale.

15. This problem is the same as problem 14 except that there are 12 balls, one of which is either heavier or lighter than the other eleven. The requirement is to pick out the odd ball in three balancings and tell whether the ball is heavier or lighter.

16. Baseball logic: Nine men—Brown, White, Adams, Miller, Green, Hunter, Knight, Smith, and Jones—play the several positions on a baseball team. Determine from the following information the position played by each man:
   a. Brown and Smith each won \$10 playing poker with the pitcher;
   b. Hunter is taller than Knight and shorter than White but each weighs more than the first baseman;
   c. the third baseman lives across the corridor from Jones in the same apartment house;
   d. Miller and the outfielders play bridge in their spare time;
   e. White, Miller, Brown, the right fielder, and the center fielder are bachelors and the rest are married;
   f. of Adams and Knight, one plays an outfield position;
   g. the right fielder is shorter than the center fielder;
   h. the third baseman is a brother of the pitcher's wife;
   i. Green is taller than the infielders and the battery, except for Jones, Smith, and Adams;
   j. the second baseman beat Jones, Brown, Hunter, and the catcher at cards;
   k. the third baseman, the shortstop, and Hunter made \$150 each speculating in General Motors stock;
   l. the second baseman is engaged to Miller's sister;
   m. Adams lives in the same house as his own sister but dislikes the catcher;
   n. Adams, Brown, and the shortstop lost \$200 each speculating in grain;
   o. the catcher has three daughters, the third baseman has two sons, but Green is being sued for divorce.

Problems such as these do much to make the mathematics class more enjoyable for both the student and the teacher, as well as providing challenging problems for the students who most need to meet such a challenge.

A school administrator once told me that he could tell how good a teacher was simply by standing outside the classroom door at dismissal time and listening to the students as they left the classroom.

What do your students talk about as they leave your room?

Do they talk about a tough problem that has "stumped" them all?

Making the mathematics classroom attractive

by Irwin N. Sokol, Collinwood High School, Cleveland, Ohio

What sets off a mathematics classroom as being different from any other classroom? Unfortunately, one identifying characteristic seems to be that in so many mathematics classrooms there is nothing to look at but the same bleak, drab, monotonous walls day after day. Such walls have become almost a stereotype. Typical reasons given for the lack of an attractive atmosphere in a mathematics classroom are that mathematics does not lend itself to the creation of an interesting atmosphere or that there are no commercial materials that we could exhibit, in contrast to the great number adaptable to science, social studies, and English.

The purposes of this article are to show some ways to make the mathematics classroom more attractive and to give sources of free and inexpensive materials useful in a mathematics class.

Many local as well as national concerns will provide free or at a modest cost attractive charts, posters, pictures, articles, pamphlets, and samples that make excellent and appropriate displays. For example, for a unit on measurement you can obtain free from The Ford Motor Company, Education Department, Dearborn, Michigan, large-size prints of early means of measure and a small booklet called "How Long Is a Rod?"

Lufkin Rule Company, 1730 Hess Ave., Saginaw, Michigan, has a comic-book style pamphlet in color, "The Amazing Story of Measurement" (10¢), and several large charts which illustrate the many tools of measurement (free).

General Motors Corporation, Educational Relations, Department of Public Relations, 3044 West Grand Blvd., Detroit 2, Michigan, offers a free booklet, "Precision—A Measure of Progress," containing up-to-date information on the very fine measuring devices.

Brown and Sharpe Mfg. Company, Providence, R. I., will send without cost some giant charts showing micrometers and their parts, in addition to many pamphlets telling how to use these instruments.

The U. S. Government Printing Office, Department of Public Documents, Washington, D. C., will send you a life-size chart (B.S. Misc. Pub. No. 3), "The International Metric System," showing the various types of metric and English measures, at a cost of 50¢.

The American Can Company, Home Economics Section, 100 Park Ave., New York 17, N. Y., has a set of different size cans labeled according to their respective capacities and uses, available without charge.

For help in planning bulletin board displays, write the National Council of Teachers of Mathematics for the excellent pamphlet, "How to Use Your Bulletin Board" (50¢).

With a little ingenuity you can locate other sources also. Usually, only a minute, a sheet of school stationery, and a four-cent stamp are all that separate the "come in, you're welcome" classroom from the all too common "aw, math again" room.

No mathematics classroom is complete without the many models used daily in our work. They can be bought commercially at moderate cost or they can be student-made.

I once thought that student-produced models necessarily had to be mediocre, until I let my boys and girls try to make some. Without my having to exert very much pressure, they produced some near-professional models. Among the best were

the five Platonic solids, made of wood, nuts and bolts; and a model of Cavalieri's solids, constructed by equating the areas of fifty rectangles and fifty circles made of a soft plastic material. They were measured, cut, colored, and then mounted on a large board, even equipped with handles for easy carrying. These were shown at the Cleveland meeting of the NCTM, held in April, 1958. The hardware, the other materials, the careful fitting of each piece, the involved computations, and the almost perfect execution of the task assumed were most enlightening. Even more surprising was the fact that those who did the best work on the model construction projects were not necessarily the better students, but more often only the average in scholastic ability.

With proper encouragement and friendly guidance, your pupils can do much to make the mathematics classroom more attractive. Possibly it will be a display of items built or collected by students, such as: simple hollow pyramids and prisms; cones and cylinders made of cardboard, wood, metal, paper, or plastic; fascinating mobiles of many different and imaginative types; sewn or knitted designs; notebooks with clever covers and packed with clippings, drawings, and diagrams (also the sources from which the information was gathered); a collection of cartoons dealing with mathematics or school in general. I sometimes sprinkle these around the room to get my pupils to look at the other displays. In addition, I sometimes post perfect test papers and neat homework papers.

Shake up your imagination a little; spend a little time to change the scenery often, so that it won't get stale. Let the pupils walk around the room from time to time to look at and touch the displays. You are sure to find that your pupils like the new atmosphere.

(If you would care to send me twenty cents to cover postage and printing expenses, I will send you a copy of the list I helped prepare for the April, 1958 meeting of The National Council of Teachers of Mathematics in Cleveland, Ohio. This list contains more than 100 companies that will provide free, or at a very modest charge, many valuable aids to teaching mathematics on the elementary and secondary levels. Address requests to: Irwin N. Sokol, Collinwood High School, 15210 St. Clair Avenue, Cleveland 10, Ohio.)

What's new?

BOOKS

College

Mathematics for Engineers, Part II (5th ed.), W. N. Rose. London: Chapman and Hall, Ltd., 1958. Cloth, xii+403 pp., price 25s.

Mathematics for Higher National Certificate (Vol. 2, Electrical), S. W. Bell and H. Matley, New York: Cambridge University Press, 1958. Cloth, x+486 pp., $6.50.

Miscellaneous

Mathematics in Fun and in Earnest, Nathan Altshiller Court. New York: The Dial Press, 1958. Cloth, 250 pp., $4.75.

INSTRUCTIONAL MATERIALS

Mathematics Charts, L. E. Christman, Yorkville, Illinois. Set of twelve charts in black and white dealing with the history of mathematics from the dawn of history to 1950; $1.50 per set.

Miniature Slated Globe (Model 590), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. 8-inch globe with black slated surface in cradle mount with equatorial ring suitable for use by students; $10.00 each.

A New Look at Budgeting, Money Management Institute, Household Finance Corporation, Prudential Plaza, Chicago 1, Illinois. Color filmstrip with script; loan for one week, free.

Things of Science: Kit #209—Computation.

Things of Science: Kit #213—Curves, Science Service, 1719 N Street, N.W., Washington 6, D. C. Kits, 75¢ each or 3 for $1.50.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Nominating Committee

The Committee on Nominations and Elections presents its slate of nominees for offices to be filled in the 1959 election. The term of office for the two vice-presidents is two years. Three directors are to be elected for terms of three years.

In making nominations for the three director positions, the Committee followed the directive adopted by the Board of Directors in 1955 which states, "Nominations shall be made so that there shall be not more than one director elected from each state, and that there shall be one director, and not more than two, elected from each region." Members may consult THE MATHEMATICS TEACHER for October, 1955, for a map of the regions as they are now defined.

Ballots will be mailed on or before February 10, 1959 from the Washington Office to members of record as of that date. Ballots returned and postmarked not later than March 10, 1959 will be counted.

The Committee wishes to thank the many members of the NCTM for help in giving their suggestions for nominees. It is hoped that all members of our organization will be sure to exercise their privilege of voting.

MILTON BECKMANN, *Chairman*
CLIFFORD BELL
CHARLES BUTLER
ROBERT FOUCH
MARTHA HILDEBRANDT
MILDRED KEIFFER
ANN PETERS
MYRON ROSSKOPF
MARIE WILCOX

NOMINEES FOR VICE-PRESIDENT—COLLEGE LEVEL

PHILLIP S. JONES

Z. L. LOFLIN

Phillip S. Jones

Professor of Mathematics and Education, University of Michigan, Ann Arbor, Michigan.

A.B., Ph.D., University of Michigan.

Jackson Junior High School, Jackson, Michigan; Edison Institute of Technology, Dearborn, Michigan; University School, Ohio State University; Duke University.

Member: NCTM; AMS; MAA; Phi Rho Pi; Sigma Xi; Phi Kappa Phi; and AAAS.

Activities in NCTM: Member, Board of Directors; Departmental Editor, THE MATHEMATICS TEACHER; Editor, *Twenty-fourth Yearbook;* former Associate Editor, THE MATHEMATICS TEACHER.

Other activities: Member, NAS-NRC Committee; SMSG Advisory Committee; Mathematics Committee, C.E.E.B.; formerly member of Board of Governors of MAA.

Publications: Numerous articles in THE MATHEMATICS TEACHER, *The Monthly*, and other magazines.

Z. L. Loflin

Professor and Head Mathematics Department, Southwestern Louisiana Institute.

B. S., M. S., Louisiana State University; Ph.D., Columbia University.

Member: NCTM; Phi Kappa Phi; Pi Mu Epsilon; Kappa Mu Epsilon; Louisiana Teachers Association; NEA; American Educational Research Association; American Mathematical Society; American Chemical Society; AAUP; American Institute of Mining and Metallurgical Engineers; American Association for Engineering Education; Central Association of Science and Mathematics Teachers; Rotary Club of Lafayette. Listed in *American Men of Science* and *Who's Who in the South and Southwest.*

Activities in NCTM: Editorial Board, THE MATHEMATICS TEACHER.

Other activities: Member, National Board of Governors of Mathematical Association of America; Past National President, Theta Xi.

NOMINEES FOR VICE-PRESIDENT—JUNIOR HIGH SCHOOL LEVEL

MARIAN C. CLIFFE

MILDRED B. COLE

Marian C. Cliffe

Supervisor of Mathematics Curriculum, Junior and Senior High Schools, Los Angeles City Board of Education.

B. A., University of California at Los Angeles; M.A., Ed.D., University of Southern California.

Teacher of mathematics, junior and senior high schools, Los Angeles city;

evening instructor, Los Angeles City College; chairman mathematics department, Verdugo Hills Six-Year High School; junior high school counselor, Verdugo Hills High School; co-ordinator, director of summer workshops for teachers of mathematics, Los Angeles city schools; consultant for mathematics workshops, institute speaker in various districts.

Member: NCTM; California Mathematics Council; Council of Directors and Supervisors; American Association for the Advancement of Science; Texas Council of Teachers of Mathematics; Los Angeles City Teachers Mathematics Association; Society of Delta Epsilon; Pi Mu Epsilon; Pi Lambda Theta; Delta Kappa Gamma.

Activities in NCTM: Membership Committee, 1955——; Committee on Secondary School Standards, 1956–58; Cochairman of Local Arrangements, Sixteenth Summer meeting, 1956; Treasurer, Christmas meeting, 1953; on NCTM convention program at Los Angeles, Boston, Milwaukee, and Indiana University.

Other activities: State President, California Mathematics Council, 1957–59; President, Southern Section, CMC, 1955–57; Treasurer, Southern Section, CMC, 1953–55; Member, Board of Directors, CMC, 1951–53; Chairman, Subcommittee on Secondary Mathematics for Non-College Preparatory Students, California Subcommittee on Content and Sequence of Mathematics from the Ninth Through Fourteenth Grades; Delegate, Industry-Education Conference, 1957 and 1958; Secretary, Society of Delta Epsilon (Doctoral Society USC), 1958–59; Treasurer, Delta Epsilon, 1957–58; Faculty President, Verdugo Hills High School, 1951–52; Planning Committee, California Conference for Teachers of Mathematics, 1953–58; California State Scholarship Committee, Delta Kappa Gamma, 1957–59; Chairman, Curriculum Branch Staff, Los Angeles City Board of Education, 1957–58; Planning Committee, Annual Awards Convocation, U. S. C. School of Education; Group Chairman, Future Teachers' Day; Los Angeles County Fair Committee for Mathematics; Mathematics specialist on Planning and Production Committee, CBS Television series *Learning*, 1957; Reviewing Committee, Report of Commission on Mathematics, C.E.E.B.

Publications: "The Place of Evaluation in the Secondary School Program" (THE MATHEMATICS TEACHER); "Follow-up Data with Implications for Guidance" (*California Journal of Educational Research*); "Notes on the 36th Annual Convention of NCTM," "Challenge of a New School Year" (*California Mathematics Council Bulletin*); "Mathematics Evaluation in a Large City, Los Angeles" (to be published in *Bulletin of the National Association of Secondary School Principals*). Responsible for Los Angeles City Schools publications: "Outline Course of Study for Academic Mathematics," "Instructional Guide for Junior High School Mathematics," "Instructional Guide for High School Mathematics I," "Instructional Guide for High School Mathematics II."

Mildred B. Cole

Mathematics teacher, K. D. Waldo Junior High School, Aurora, Illinois; Instructor, Aurora College: evening classes, "Foundations of Arithmetic" and "Teaching of Arithmetic in the Elementary School."

B.S., University of Illinois; M.S., University of Wisconsin; additional graduate study, University of Colorado.

Elementary teacher, Harvard, Illinois; arithmetic teacher, Grades 5, 6, 7, C. M. Bardwell, Aurora; mathematics teacher, K. D. Waldo Junior High School, Aurora; Laboratory School, University of Wisconsin, Summer 1949.

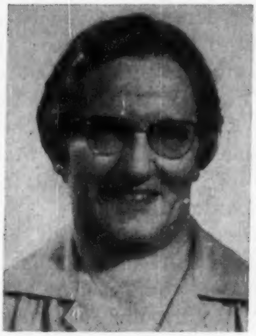
Member: NCTM; Illinois Council of Teachers of Mathematics; MAA; NEA; Illinois Education Association; AAUW.

Activities: Member, Mathematics Study Group of Allerton House Conference on Education, 1953–58; ICTM Dele-

gate to Steering Committee of Illinois Curriculum Program, 1955——; Member, Writing Committee of Three for new Illinois Mathematics Curriculum Bulletin, Kdg. 8, "Thinking in the Language of Mathematics"; President, Illinois Council of Teachers of Mathematics, 1953–54; numerous speeches and papers at mathematics conferences and teachers' institutes.

Publications: "Teaching Materials as Keys to Understanding Arithmetic in the Primary Grades"; "Arithmetic Aids for the Middle Grades."

NOMINEES FOR THE BOARD OF DIRECTORS



CAROL V. MCCAMMAN

PHILIP PEAK

CHRISTINE POINDEXTER

OSCAR SCHAAF

HENRY VAN ENGEN

Carol V. McCamman

Southeastern Region

Teacher of Mathematics, Calvin Coolidge High School, Washington, D.C.

A.B., M.A., University of California, Berkeley; further graduate work, University of California, University of Chicago, University of Michigan; NSF Institute, University of California, Los Angeles, Summer, 1957.

Examiner in the physical sciences, University of Chicago, 1931–33; research assistant, The Psychological Institute, Washington, D.C., 1933–36; high school mathematics teacher, District of Columbia Public Schools, 1936——; part-time teaching, American University, Catholic University, George Washington University; visiting associate in mathematics, Educational Testing Service, Summer, 1956.

Member: NCTM; MAA; AAAS; NEA;

AAUW; Phi Beta Kappa; Pi Mu Epsilon; Delta Kappa Gamma.

Activities in NCTM: President, District of Columbia Teachers of Mathematics (Affiliated Group, NCTM); Registration Chairman, 1955 NCTM Convention in Washington; Liaison Representative of NCTM on Joint Commission on the Education of Teachers of Science and Mathematics of the AAAS and the AACTE; attendance at conventions and participation in convention programs.

Other activities: Past President, High School Teachers Association, Washington, D.C.; High School Mathematics Contest Committee of D.C.-Maryland-Virginia Section, MAA; Member, department and city textbook and curriculum committees; City-wide Algebra Test Committee; Consultant, Committee on Affiliation, Catholic University; participant, NEA Invitational Conference on the Academically Talented, 1958.

Publications: Article in *Eighteenth Yearbook* of NCTM; assisted in revision of plane geometry text; section on education of the mathematically talented in October, 1958, *NEA Journal;* articles in local educational publications.

Philip Peak

Central Region

Assistant Dean and Associate Professor of Education, University of Indiana, Bloomington, Indiana.

B.A. in Mathematics, Iowa State Teachers College, 1930; M.S. in Mathematics, University of Iowa, 1935; Ph.D. in Mathematics and Education, Indiana University, 1955.

Mathematics and Agriculture teacher, Mechanicsville High School, 1930–35; Head of Mathematics Department, Pierre, South Dakota High School, 1935–38; Assistant Professor Mathematics, Nebraska State Teachers College, Chadron, Nebraska, 1938–42; Head of Mathematics Department of University School and Instructor of Education, Indiana University, 1942–50; Assistant Director of Student Teaching, Indiana University, 1950–56.

Member: NCTM; Central Association of Science and Mathematics Teachers; American Mathematics Association; Sigma Xi; Indiana State Teachers Association; Indiana Academy of Science; Indiana Council of Teachers of Mathematics; NEA; Phi Delta Kappa.

Activities in NCTM: Director of NCTM; Member, Executive Committee; Member, Editorial Committee of THE MATHEMATICS TEACHER; Member *Twenty-second Yearbook* Committee; former member, Film Committee; Local Chairman, Summer meeting, 1955; State Representative for National Council; former member, Board of Directors.

Other activities: Vice President and President of the Central Association of Science and Mathematics Teachers.

Publications: Contributed articles to THE MATHEMATICS TEACHER since 1946; contributed articles to *School Science and Mathematics;* reviewed books for both publications; "Research Before Writing" (*School and Society*, 1948); "Indiana University Chapter, The Society of Sigma Xi" (*American Scientist*, 1956); coauthor, How-To-Pamphlet, "How to Use Film and Film Strips"; coauthor, National Council flyer, "As We See It."

Christine Poindexter

Southwestern Region

Chairman, Department of Mathematics, Central High School, Little Rock, Arkansas.

B.S.E., Arkansas Teachers College; M.A., University of Missouri; additional work, University of Texas; participant in National Science Foundation Institute for High School Mathematics Teachers, Indiana University, 1957.

Teacher of mathematics in high schools of Arkansas for more than 30 years, the past 16 years in Central High School; Chairman of Mathematics Department for ten years; teacher, Little Rock Junior

College and Henderson Teachers College in summer sessions, Little Rock University in night school, 1958.

Member: NCTM; Arkansas Council of Mathematics Teachers; Arkansas Education Association; NEA (Life Member); Delta Kappa Gamma Society.

Activities: President, Arkansas Council of Mathematics Teachers, 1956–58; Consultant in the Teaching of Geometry at University of Arkansas Mathematics Workshops, 1953 and 1954; Sponsor of Mathematics Exhibit at annual convention of Arkansas Education Association for several years; Past President, Little Rock Education Council, 1949–51; Delegate to National Citizenship Conference, Washington, D. C., 1952; Past President of Mathematics Section of A.E.A., before which many talks have been made; Member, Committee on Preparation of the Mathematics Teacher under grant from Ford Foundation.

Oscar F. Schaaf

Western Region

Head, Department of Mathematics, South Eugene High School, and Assistant Professor of Education, University of Oregon, Eugene, Oregon.

B.A., University of Wichita, Wichita, Kansas; M.A., University of Chicago; Ph.D., Ohio State University.

Teacher of Mathematics, Leoti, Kansas, and Anthony, Kansas; Instructor in Education, Ohio State University; Teacher of Mathematics at University School, Ohio State University; Instructor in summer session, Department of Education, Ohio State University; Counselor, Science Teaching Improvement Program, American Association for the Advancement of Science; Assistant Professor in summer sessions, Department of Education, University of Oregon.

Member: NCTM; Oregon Council of Teachers of Mathematics; Ohio Council of Teachers of Mathematics (charter member).

Activities in NCTM: Associate Editor of present *Mathematics Student Journal;* present State Representative for NCTM in Oregon; appearances on convention programs.

Other activities: Secretary and Vice President of Ohio Council of Teachers of Mathematics; Past President of Oregon Council of Teachers of Mathematics; Mathematics Curriculum Consultant for Eugene Public Schools and several other Oregon school systems.

Publications: Consultant and contributor to forthcoming *Handbook for Oregon Secondary School Mathematics Teachers;* Advisor and Editor of revision of the *Oregon Mathematics Scope and Sequence;* many articles in professional periodicals.

Henry Van Engen

North Central Region

Professor of Education and Mathematics, University of Wisconsin, Madison, Wisconsin.

A.B., Nebraska Wesleyan University; Ph.D., University of Michigan.

Elementary, Junior High School, and Senior High Schools, Nebraska, Michigan, and Ohio; Western Reserve University; Kansas State University; Iowa State Teachers College.

Member: NCTM; MAA; AMS; Sigma Xi; Phi Kappa Phi; Phi Beta Kappa; Pi Mu Epsilon; Kappa Mu Epsilon.

Activities in NCTM: Editor, THE MATHEMATICS TEACHER; Member, Board of NCTM; Member, Elementary Curriculum Committee.

Other activities: Past President, Kappa Mu Epsilon; Member, SMSG Advisory Committee, Commission on Mathematics.

Publications: Numerous articles appearing in THE MATHEMATICS TEACHER, *School and Society, School Science and Mathematics, The Arithmetic Teacher,* and other magazines; chapters in various yearbooks; coauthor of textbooks.

Overseas teaching posts

Foreign teaching posts will be available in Army-operated schools for American children in Germany, France, Italy, Japan, and Okinawa for the 1959–60 school year. The greatest number of vacancies will be for elementary teachers experienced in the primary grades. Secondary teachers who qualify in two major fields will be needed also. Opportunities generally exist for school librarians, guidance counselors, and dormitory supervisors. A limited number of administrative positions are expected.

QUALIFICATIONS include a bachelor's degree, teacher training, and two years' experience.

Government transportation is furnished and rent-free living quarters are available in most areas. Salary for the instructional staff is $415 monthly. The tour of duty is one year.

To assure consideration for the coming school year, inquiry regarding application procedure should be made immediately to:

Teacher Recruitment
Department of the Army, DCSPER
Office of Civilian Personnel, OAD
Interchange & Recruitment Coord. Branch,
Old Post Office Building
12th and Pennsylvania Ave., N. W.,
Washington 25, D. C.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

THIRTY-SEVENTH ANNUAL MEETING

April 1–4, 1959
Baker Hotel, Dallas, Texas
Arthur W. Harris, 4701 Cole Avenue, Dallas 5, Texas

JOINT MEETING WITH NEA

June 29, 1959
St. Louis, Missouri
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NINETEENTH SUMMER MEETING

August 17–19, 1959
University of Michigan, Ann Arbor, Michigan
Phillip S. Jones, Mathematics Department, University of Michigan, Ann Arbor, Michigan

Other professional dates

Forty-second Annual Meeting, Mathematical Association of America

January 22–23, 1959
University of Pennsylvania, Philadelphia, Pennsylvania
Harry M. Gehman, University of Buffalo, Buffalo 14, New York

Mathematics Section of the New York Society for the Experimental Study of Education

February 14; March 14; April 17, 1959
Teachers College, Columbia University, New York, New York
John A. Schumaker, Montclair State College, Montclair, New Jersey

Illinois Council of Teachers of Mathematics

March 7, 1959, Greenville, Illinois
March 28, 1959, Macomb, Illinois
April 11, 1959, Normal Illinois
April 15, 1959, Charleston, Illinois
April 18, 1959, Arlington Heights, Illinois
April 19, 1959, Carbondale, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

Chicago Elementary Teachers' Mathematics Club

March 9, 1959
Toffenetti's Restaurant, 65 W. Monroe Street, Chicago, Illinois
Romana H. Goldblatt, Burley School, Chicago, Illinois